MATH 401: Lecture 2 (08/21/2025)

Today: * sets and operations

Sets and Operations (LS IRA 1.2)

Set: Collection of mathematical objects.

They can be finite, e.g., $\{2,5,9,1,6\}$, or infinite, e.g., $[0,1]$, the collection of all $x \in \mathbb{R}$ with $0 \leq x \leq 1$.

"element of" set of all real numbers

Given sets A, B we have

$A \subseteq B$: A is a subset of, or equal to, B.

$A \subset B$: A is a strict subset of B, i.e., there is at least one $x \in B$ such that $x \notin A$.
But $\forall x \in A$, $x \in B$ holds.

To prove $A = B$, we often prove $A \subseteq B$ and $A \supseteq B$ (or $B \subseteq A$).

Here are ~~some~~ standard sets we will use regularly.

ϕ: empty set.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, set of all natural numbers

$\mathbb{R}$ = set of all real numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, set of all integers

$\mathbb{Q}$ = set of rational numbers, $\mathbb{C}$ = set of complex numbers.

$\mathbb{R}^n$: set of all real n-tuples, or n-vectors

Notation for sets: $[-2, 1] = \{x \in \mathbb{R} \mid -2 \leq x \leq 1\}$.

closed interval from -2 to 1

"such that"
could also use ":" instead of "|".

More generally, $A = \{a \in B \mid P(a)\}$.

bigger set than A property

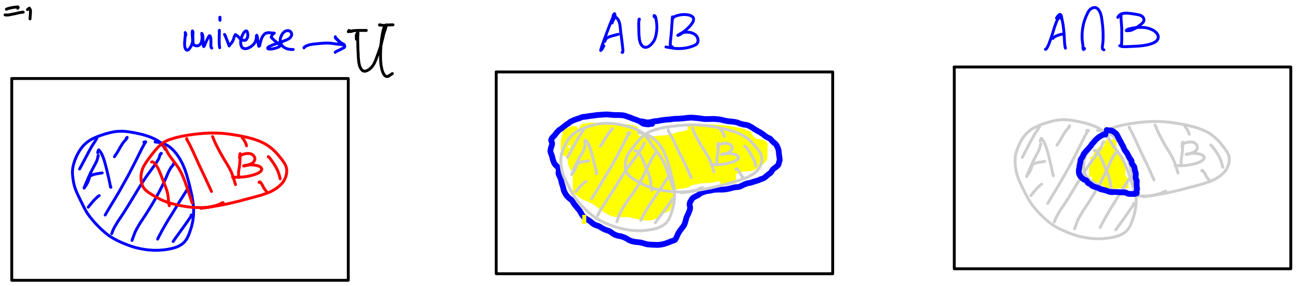
Union and Intersection

If A_i are sets for $i=1,\ldots,n$, i.e., $A_1, A_2, \ldots, A_n$ are sets, then

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{a \mid a \in A_i$ for at least one $i\}$ is their union,

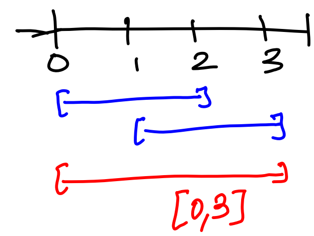
"for all"

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{a \mid a \in A_i \ \forall i\}$ is their intersection.



LSIRA 1.2 Prob1

Show $[0,2] \cup [1,3] = [0,3]$.



We show $[0,2] \cup [1,3] \subseteq [0,3]$ and
$[0,2] \cup [1,3] \supseteq [0,3]$.

($\subseteq$) Let $x \in [0,2] \cup [1,3]$

$\Rightarrow x \in [0,2]$ or $x \in [1,3]$ (definition of $\cup$).

$x \in [0,2] \Rightarrow x \in [0,3]$ (as $[0,3]$ contains $[0,2]$)

$x \in [1,3] \Rightarrow x \in [0,3]$. In either case, $x \in [0,3]$.

Hence $[0,2] \cup [1,3] \subseteq [0,3]$.

($\supseteq$) Let $x \in [0,3]$. Hence $0 \le x \le 3$. Then we get that either $x \le 2$, and hence $x \in [0,2]$, or $x \in (2,3]$.

But if $x \in (2,3]$ then $x \in [1,3]$ (as $[1,3]$ includes $(2,3]$).

$\Rightarrow x \in [0,2] \cup [1,3]$.

Hence $[0,3] \subseteq [0,2] \cup [1,3]$. □

The result is an obvious one. But we go through the steps of a formal proof more for practice!

Distributive Laws of Union and Intersection

For all sets $B, A_1, \cdots, A_n$, we have

LSIRA (1.2.1) $$B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n).$$

Using more compact notation, we can write

$$B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (B \cap A_i)$$

Proof

We will prove

$$B \cap (A_1 \cup \cdots \cup A_n) \subseteq (B \cap A_1) \cup \cdots \cup (B \cap A_n), \text{ and}$$

$$B \cap (A_1 \cup \cdots \cup A_n) \supseteq (B \cap A_1) \cup \cdots \cup (B \cap A_n).$$

('$\subseteq$') Let $x \in B \cap (A_1 \cup \cdots \cup A_n)$.

$\Rightarrow x \in B$ and $x \in (A_1 \cup \cdots \cup A_n)$ (definition of $\cap$)

$\Rightarrow x \in B$ and $x \in A_i$ for at least one A_i. (defn. of $\cup$)

$\Rightarrow x \in B \cap A_i$ for at least one A_i.

$\Rightarrow x \in (B \cap A_1) \cup \cdots \cup (B \cap A_n)$.

('$\supseteq$') Let $x \in (B \cap A_1) \cup \cdots \cup (B \cap A_n)$.

$\Rightarrow x \in (B \cap A_i)$ for at least one A_i.

$\Rightarrow x \in B$ and $\underbrace{x \in A_i \text{ for at least one } A_i}$

$\Rightarrow x \in B$ and $x \in (A_1 \cup \cdots \cup A_n)$

$\Rightarrow x \in B \cap (A_1 \cup \cdots \cup A_n)$. $\square$

LSIRA (1.2.2) is assigned in Homework 1.

Set Difference and Complement

We write $A \setminus B$ or $A - B$ "setminus"

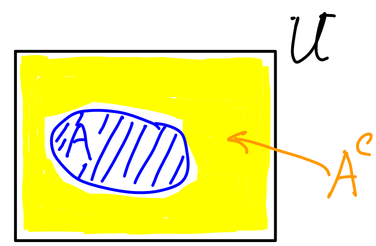
Caution! $* \; A\setminus B \neq B \setminus A$!

"A setminus B" is $A \setminus B = \{a \mid a \in A, a \notin B\}$.

If $\mathcal{U}$ is the universe, i.e., $A \subseteq \mathcal{U}$ for all sets A, then

$$A^c = \mathcal{U} \setminus A = \{a \in \mathcal{U} \mid a \notin A\}$$ is the

complement of A (or A-complement).



De Morgan's Laws

LSIRA (1.2.3) $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$.

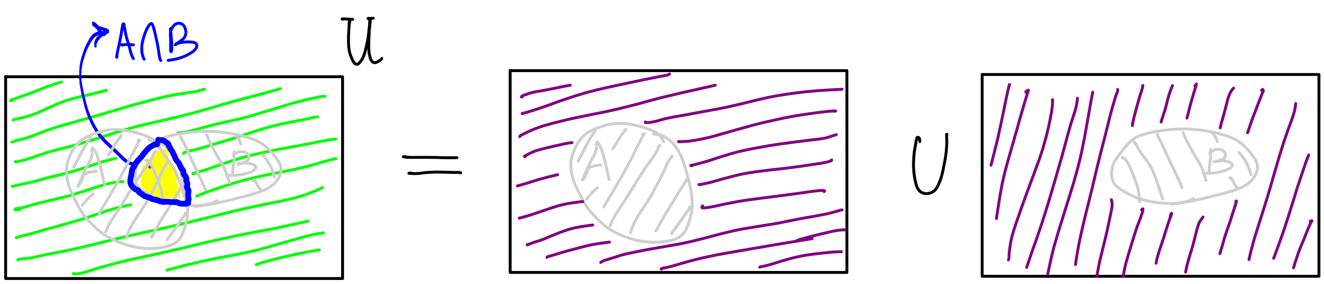
"complement of union = intersection of complements"

LSIRA (1.2.4) $(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c$.

complement of intersection = union of complements.

See LSIRA for the proof.

Let's illustrate (1.2.4) for $n=2$, i.e., with A_1 and A_2 first.



$(A \cap B)^c \qquad A^c \qquad B^c$

We will prove subset inclusion in both directions.

($\subseteq$) Let $x \in (A_1 \cap \cdots \cap A_n)^c$.

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$ (definition of complement)

$\Rightarrow x \notin A_j$ for some j. (definition of $\cap$)

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \in A_1^c \cup \cdots \cup A_n^c$.

Hence $(A_1 \cap \cdots \cap A_n)^c \subseteq A_1^c \cup \cdots \cup A_n^c$.

($\supseteq$) Let $x \in A_1^c \cup \cdots \cup A_n^c$.

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \notin A_j$ for some j

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$. since $x \notin A_j$ for some j, it cannot be in the intersection of all A_i's.

$\Rightarrow x \in (A_1 \cap \cdots \cap A_n)^c$.

Hence $A_1^c \cup \cdots \cup A_n^c \subseteq (A_1 \cap \cdots \cap A_n)^c$. $\square$

Cartesian Products

For A, B : sets, we define

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$ → cartesian product of A and B

given A_i, $i = 1, \ldots, n$ $(A_1, \ldots, A_n)$, we define

→ compact notation $\prod$: product

$$A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \mid \underbrace{a_i \in A_i \ \forall i}_{a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n}\}.$$

e.g., $\mathbb{R}^n$: set of n-tuples of real numbers (or set of real n-vectors)

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

We'll finish the proof in the next lecture...