MATH401: Lecture 3 (08/26/2025)

Today: * families of sets, properties
* functions, images, pre images

We first do a problem on Cartesian products...

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

'$\subseteq$' Let $(x,y) \in (A \cup B) \times C$.

$\Rightarrow x \in A \cup B,\ y \in C$ (Definition of cartesian product)

$\Rightarrow x \in A$ or $x \in B$, $y \in C$

If $x \in A$ then $(x,y) \in A \times C$, and

if $x \in B$ then $(x,y) \in B \times C$.

$\Rightarrow (x,y) \in A \times C$ or $(x,y) \in B \times C$

$\Rightarrow (x,y) \in (A \times C) \cup (B \times C)$.

'$\supseteq$' Let $(x,y) \in (A \times C) \cup (B \times C)$

$\Rightarrow (x,y) \in A \times C$ or $(x,y) \in B \times C$

$\Rightarrow x \in A,\ y \in C$ or $x \in B,\ y \in C \Rightarrow (x \in A$ or $x \in B),\ y \in C$.

$\Rightarrow x \in A \cup B,\ y \in C \Rightarrow (x,y) \in (A \cup B) \times C$.

$\square$

LSIRA 1.3 Families of Sets

Recall: $B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (B \cap A_i)$. → compact notation for distributive law (from Lecture 2)

We could write, instead, $B \cap \left(\bigcup_{i \in \mathcal{I}} A_i \right) = \bigcup_{i \in \mathcal{I}} (B \cap A_i)$, where $\mathcal{I} = \{1, 2, \ldots, n\}$.

We now generalize $\mathcal{I}$ to be infinite in some cases, or indexing more general collections in general.

Def A collection of sets is a **family**.

e.g., $\mathcal{A} = \{ [a,b] \mid a, b \in \mathbb{R} \}$ is the family of all closed intervals on $\mathbb{R}$.

Union and Intersection

We extend union, intersection, as well as their distribution to families.

$$\bigcup_{A \in \mathcal{A}} A = \{ a \mid a \in A \text{ for some } A \in \mathcal{A} \}.$$

→ collection of elements that belong to at least one set in the family

$$\bigcap_{A \in \mathcal{A}} A = \{ a \mid a \in A \text{ for all } A \in \mathcal{A} \}$$

→ collection of elements that belong to every set in the family.

We naturally extend distributive and De Morgan's laws to families . .

$$B \cap \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (B \cap A), \quad \left(\bigcap_{A \in \mathcal{A}} A \right)^c = \bigcup_{A \in \mathcal{A}} A^c, \text{ etc.}$$

We now work on some problems involving families of sets.

LSIRA 1.3 Prob 1 (Pg 12)

Show that $\bigcup_{n\in\mathbb{N}} [-n,n] = \mathbb{R}$.

('$\subseteq$') $\mathbb{R}$ is the universe here, so $\bigcup_{n\in\mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

Or, observe that $[-n,n] \in \mathbb{R}$ for each $n\in\mathbb{N}$, hence $\bigcup_{n\in\mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

('$\supseteq$') Let $x\in\mathbb{R}$. To be more careful, we could consider $x=0$ separately. Note that $x=0 \in [-n,n]\ \forall n\in\mathbb{N}$.

Let $m = \lceil |x| \rceil$, ceiling of absolute value of x, i.e., the smallest natural number $\geq |x|$.

$\lceil x \rceil = \text{ceil}(x) =$ smallest integer $\geq x$.

Then $x \in [-m,m] = [-\lceil |x| \rceil, \lceil |x| \rceil]$, as

$x \leq |x| \leq \lceil |x| \rceil = m$, and $x \geq -|x| \geq -\lceil |x| \rceil$.

e.g., $x=-3.3 \Rightarrow x \geq -|-3.3| = -3.3 \geq -4$.

$\Rightarrow x \in \bigcup_{n\in\mathbb{N}} [-n,n]$.

$\square$

LSIRA 1.3 Prob 4

Show $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}] = \phi$ (empty set).

('$\supseteq$') $\phi \subseteq A$ for any set A (trivially).

('$\subseteq$') We show $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}] \subseteq \phi$.

$\phi^c = \mathbb{R}$. Hence we show $x\in\mathbb{R}$ is not in $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

For $x\in\mathbb{R}$, we show $x \notin \bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

If $x \leq 0$, then clearly, $x \notin (0,\frac{1}{n}]\ \forall n\in\mathbb{N}$.

If $x \geq 1$, then $x \notin (0,\frac{1}{2}]$ for $n=2$, for instance.

Let $0<x<1$. Consider $m=\lceil \frac{1}{x} \rceil+1$.

Then $x \notin (0, \frac{1}{m}]$ as $x > \frac{1}{m} = \frac{1}{\lceil \frac{1}{x} \rceil+1}$. $\left(\lceil \frac{1}{x} \rceil+1 > \frac{1}{x}\right)$

$$\Rightarrow \quad x \notin \bigcap_{n\in\mathbb{N}} (0, \tfrac{1}{n}].$$

Q. Why take $\lceil \frac{1}{x} \rceil+1$?

Consider $x=\frac{1}{5}$, for instance.
Then $\lceil \frac{1}{x} \rceil = \lceil 5 \rceil = 5$.
Hence $x \in (0, \frac{1}{m}]$ here!

$\square$

<u>LSIRA 1.3 Prob 5 (Pg 12)</u>

Prove that $B \cup \left(\bigcap_{A\in\mathcal{A}} A\right) = \bigcap_{A\in\mathcal{A}} (B\cup A)$.

$(\subseteq)$ Let $x \in B \cup \left(\bigcap_{A\in\mathcal{A}} A\right)$

$\Rightarrow x \in B$ or $x \in \bigcap_{A\in\mathcal{A}} A$

$\Rightarrow x\in B$ or $x \in A$ for each $A \in \mathcal{A}$.

$\Rightarrow x \in B\cup A$ for each $A\in\mathcal{A}$.

$\Rightarrow x \in \bigcap_{A\in\mathcal{A}} (B\cup A)$.

$(\supseteq)$ Let $x \in \bigcap_{A\in\mathcal{A}} (B\cup A)$

$\Rightarrow x \in B\cup A$ for every $A\in\mathcal{A}$.

$\Rightarrow x\in B$ or $\underbrace{x \in A \text{ for every } A\in\mathcal{A}}$.

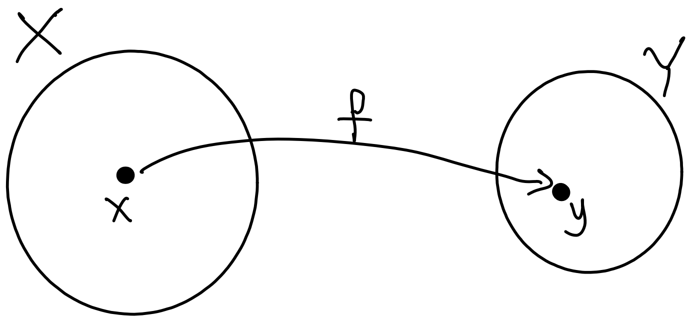
$\Rightarrow x\in B$ or $x \in \bigcap_{A\in\mathcal{A}} A \Rightarrow x \in B\cup\left(\bigcap_{A\in\mathcal{A}} A\right)$.

$\square$

LSIRA 1.4 **Functions**

A function $f: X \to Y$ for sets X, Y is a rule that assigns for each $x \in X$ a **unique** $y \in Y$. We write $f(x) = y$, or

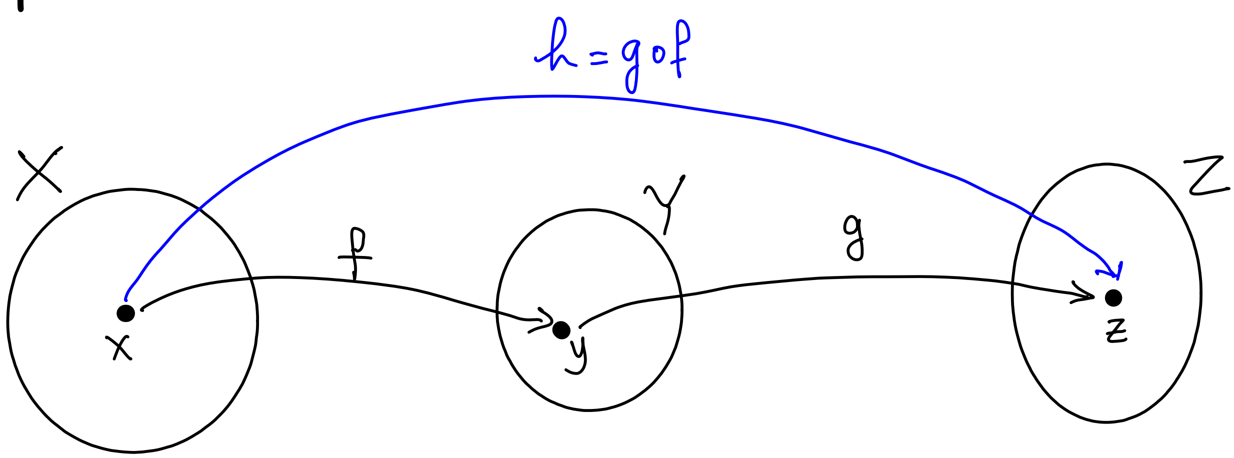
$x \mapsto y$ "maps to".



Rather than the graphs of functions you may have seen previously, we think of such visualizations for functions now.

X is the domain and Y the codomain of f.

Compositions

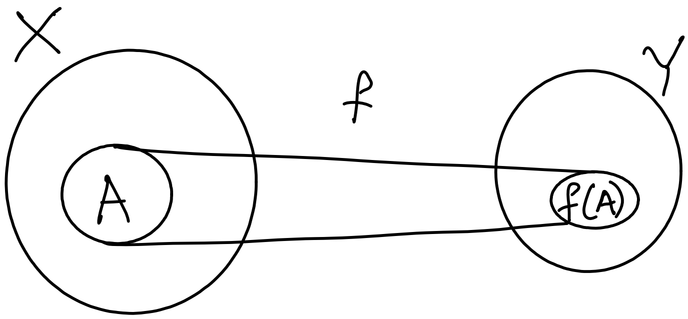


Let $f: X \to Y$ and $g: Y \to Z$ be functions. Then their composition is specified as $h: X \to Z$ defined as $h(x) = g(f(x))$.
The composition is written as $g \circ f$, with $g \circ f(x) = g(f(x))$.

"composition of f and g"

$f_1(f_2(\cdots f_n(x)))\cdots)$ → composition of n functions $f_1, f_2, \ldots, f_n$

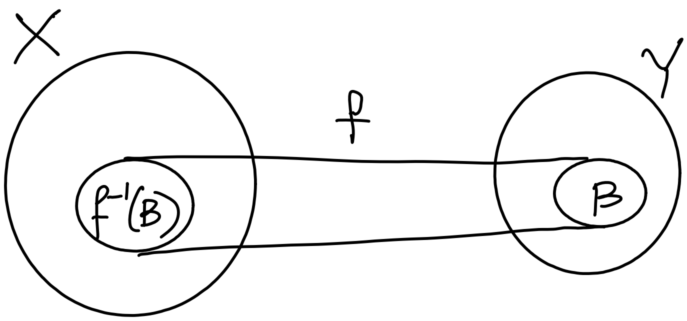
Function: $f: X \to Y$. We now define images and preimages under f.



For $A \subseteq X$, $f(A) \subseteq Y$ is defined as

$$f(A) = \{ f(a) \mid a \in A \},$$

and is called the **image** of A under f.



For $B \subseteq Y$, the set $f^{-1}(B) \subseteq X$ defined as

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}$$

is the **inverse image** or **preimage** of B under f.

In the next lecture, we consider how preimages and images commute with unions and intersections, or not...