MATH 401: Lecture 4 (08/28/2025)

Today: * images/preimages and unions/intersections
* injective/surjective functions
* relations, equivalence relations, partitions

We now consider how images and inverse images commute (or not) with unions and intersections of families of sets.

Prop 1.4.1 Inverse images commute with arbitrary unions and intersections:

$$f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) = \bigcup_{B\in\mathcal{B}} f^{-1}(B) \quad \text{and}$$

"inverse of union = union of inverses"

$$f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right) = \bigcap_{B\in\mathcal{B}} f^{-1}(B)$$

"inverse of intersection = intersection of inverses"

Proof (of the second statement) $\longrightarrow$ See LSIRA for proof of first statement

($\subseteq$) Let $x \in f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right) \Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B$

$\Rightarrow f(x) \in B$ for every $B \in \mathcal{B}$.

$\Rightarrow x \in f^{-1}(B)$ for every $B \in \mathcal{B}$.

$\Rightarrow x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$.

($\supseteq$) Let $x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$

$\Rightarrow x \in f^{-1}(B)$ for every $B \in \mathcal{B}$.

$\Rightarrow f(x) \in B$ for every $B \in \mathcal{B}$.

$\Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B \Rightarrow x \in f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right)$. $\square$

We saw that inverse images commute with unions and intersections. But forward images behave a bit differently.

<u>Prop 1.4.2</u> $f: X \to Y$ is a function, $\mathcal{A}$ is a family of subsets of X.

Then $f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f(A)$, $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

<u>Proof</u>

('$\subseteq$') Let $y \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$

"There exists"

$\Rightarrow \exists\, x \in \bigcup_{A\in\mathcal{A}} A$ such that $f(x) = y$.

$\Rightarrow x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x) = y$

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow y \in \bigcup_{A\in\mathcal{A}} f(A)$.

($\supseteq$) Let $y \in \bigcup_{A\in\mathcal{A}} f(A)$.

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow \exists\, x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x) = y$.

$\Rightarrow \exists x \in \bigcup_{A\in\mathcal{A}} A$ such that $f(x) = y$.

$\Rightarrow y \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$

LSIRA gives a slightly different proof for ($\supseteq$):

"for all"

$A \subseteq \bigcup_{A\in\mathcal{A}} A \quad \forall A \in \mathcal{A}$

$\Rightarrow f(A) \subseteq f\left(\bigcup_{A\in\mathcal{A}} A\right) \quad \forall A \in \mathcal{A}$

Since this result holds for every $A \in \mathcal{A}$, we can write

$$\bigcup f(A) \subseteq f\left(\bigcup_{A\in\mathcal{A}} A\right).$$

$\square$

We consider intersections now: $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

__Proof for ('$\subseteq$')__

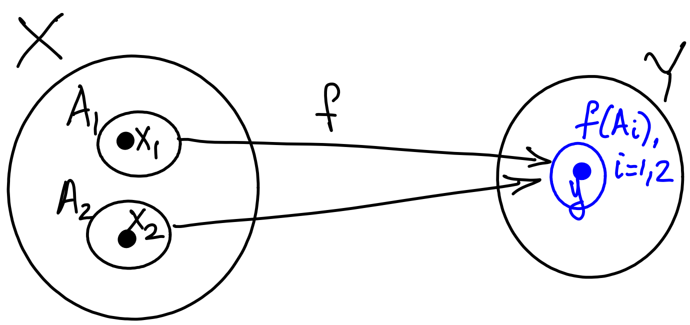
$$\bigcap_{A\in\mathcal{A}} A \subseteq A \quad \forall A\in\mathcal{A}$$

$$\Rightarrow f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq f(A) \;\forall A\in\mathcal{A}.$$

Since this inclusion holds for every $A\in\mathcal{A}$, we get

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A).$$

__Counterexample for ('$\supseteq$') for $\cap$__



For $x_1 \neq x_2$, $x_1, x_2 \in X$, let $f(x_i) = y$, $i=1,2$.

Let $A_i = \{x_i\}$, $i=1,2$. $\Rightarrow \bigcap_{i=1,2} A_i = \phi$ (empty set).

But note that $f(A_i) = \{y\}$, $i=1,2$.

$\Rightarrow f\left(\bigcap_{i=1,2} A_i\right) = \phi$. But $\bigcap_{i=1,2} f(A_i) = \{y\} \neq \phi$.

$$\Rightarrow \bigcap_{i=1,2} f(A_i) \not\subseteq f\left(\bigcap_{i=1,2} A_i\right).$$

But we get this reverse inclusion if we specify that f is injective.

Def Let $f: X \to Y$ be a function.

f is **injective** (1-to-1) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Equivalent definition: For any $y \in Y$, there is **at most** one $x \in X$ s.t. $f(x) = y$.
→ there could be no $x \in X$

f is **surjective** (onto) if for every $y \in Y$, there is **at least** one $x \in X$ such that $f(x) = y$.
→ there could be more than one

f is **bijective** if it is both injective and surjective.

LSIRA 1.4 prob 4 (pg 17)

Let $f: \underset{X}{\mathbb{R}} \to \underset{Y}{\mathbb{R}}$ be a strictly increasing function, i.e.,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for } x_i \in \mathbb{R},\ i = 1, 2.$$

1. Show that f is injective.
2. Does it have to be surjective? → Either give a proof or a counterexample.

1. We show $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
Without loss of generality (WLOG), let $x_1 < x_2$. → The same result holds when $x_2 < x_1$ as well.
Then $f(x_1) < f(x_2)$, as f is strictly increasing.
Hence $f(x_1) \neq f(x_2)$, and so f is injective.

2. No. $f = \arctan(x)$ is strictly increasing.
$f: \mathbb{R} \to \mathbb{R}$, but $\arctan(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}$.
So f need not be surjective.
Another example is $f = e^x$.

Relations (LSIRA 1.5)

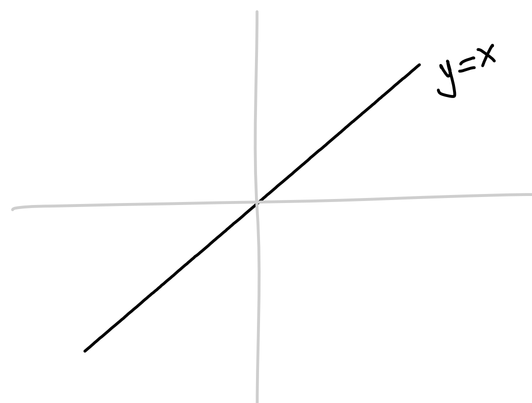
We had seen functions, where a unique $y \in Y$ is assigned for each $x \in X$ by $f: X \to Y$. But entities are related in other ways — numbers are $>$ or $<$ each other, lines are parallel, etc. We define relations formally to describe such dependencies.

Def A relation R on a set X is a subset of $X \times X$.

Cartesian product of X with itself

We write xRy, $(x,y) \in R$, or $x \sim y$.

e.g., $R = \{(x,y) \in \mathbb{R}^2 \mid x = y\}$.



Recall, $y = x$ is the 45° line through $(0,0)$. All points are "related" by them belonging to this line.

Here is another relation (on integers):

$P = \{(x,y) \in \mathbb{Z}^2 \mid x, y \text{ have same parity}\}$.

So, all odd integers are related, and so are all even integers.

Some relations have more structure than default – as defined below.

Equivalence Relations

Def A relation $\sim$ on X is an **equivalence relation** if it is

(i) reflexive, i.e., $x \sim x \ \forall x \in X$;

(ii) symmetric, i.e., $x \sim y \Rightarrow y \sim x \ \forall x, y \in X$; and

(iii) transitive, i.e., $x \sim y,\ y \sim z \Rightarrow x \sim z \ \forall x, y, z \in X$.

Note that $<$ is not reflexive, or symmetric, e.g., $5 \not< 5$, and $3 < 5 \not\Rightarrow 5 < 3$.

Def Given an equivalence relation $\sim$ on X, we define the **equivalence class** $[x]$ of $x \in X$ as
$$[x] = \{y \in X \mid x \sim y\}.$$
$\rightarrow$ the set of all "relatives" of x

The collection of equivalence classes forms a partition of X.

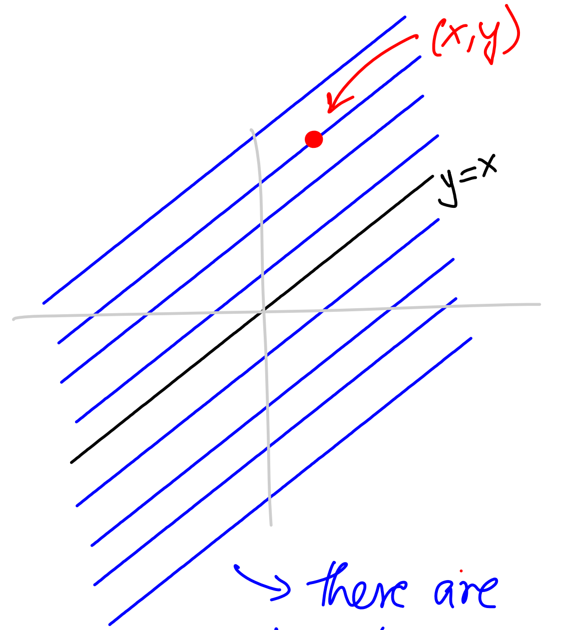
Def A **partition** $\mathcal{P}$ of X is a family of nonempty subsets of X such that $x \in X$ satisfies $x \in P \in \mathcal{P}$ for exactly one P in $\mathcal{P}$ (for every $x \in X$).

The elements P of $\mathcal{P}$ are called **partition classes** of $\mathcal{P}$.

e.g., $\mathcal{P} = \{ \underbrace{\{2k, k \in \mathbb{Z}\}}_{\text{even integers}}, \underbrace{\{2k+1, k \in \mathbb{Z}\}}_{\text{odd integers}} \}$ is a partition of $\mathbb{Z}$.

Here is a direct example of a partition of $\mathbb{R}^2$.

The collection of all lines with slope $= 1$ ($45°$) is a partition of $\mathbb{R}^2$.



Any point in $\mathbb{R}^2$ belongs to exactly one line with a slope of $m=1$ (i.e., $45°$ degree slope).

$\rightarrow$ there are infinitely many lines with slope $m=1$.

We have not checked that the defining relation is an equivalence relation, but this can be done easily.

$\rightarrow$ recall, the point-slope form of the equation of a line: $\frac{y-y_0}{x-x_0} = m$, given slope m and one point (x_0, y_0).