MATH401 : Lecture 5 (09/02/2025)

Today: * equivalence relations and partitions
* countability

Recall: * $\sim$ is an equivalence relation on X : $x\sim x$, $x\sim y \Rightarrow y\sim x$, $x\sim y, y\sim z \Rightarrow x \sim z$
* partition of X $\mathcal{P} = \{P\}$

We show that equivalence relations naturally define partitions.

Prop 1.5.3 If $\sim$ is an equivalence relation on X, then the collection of equivalence classes $\mathcal{P} = \{[x] \mid x \in X\}$ is a partition of X.

Proof We show each $x \in X$ belongs to exactly one equivalence class.

$x \sim x$ $\quad$ $\sim$ is equivalence relation, so is reflexive ((i))

$\Rightarrow x \in [x]$ $\longrightarrow$ So, each $x \in X$ belongs to at least its own class.

We now show if $x \in [y]$ for $y \in X$, $y \neq x$, then $[x] = [y]$.

We show $[x] \subseteq [y]$ and $[x] \supseteq [y]$.

($\subseteq$) Let $z \in [x]$

$\Rightarrow x \sim z$ $\quad$ Definition of $[x]$

We assumed $x \in [y] \Rightarrow y \sim x$

$\sim$ is an equivalence relation, so $\underbrace{y\sim x,\ x\sim z \Rightarrow y \sim z}_{\sim \text{ is transitive ((iii))}}$.

$\Rightarrow z \in [y]$.

($\supseteq$) let $z \in [y]$ $\quad \Rightarrow y \sim z$

Also, $x \in [y] \Rightarrow y \sim x$

$\sim$ is equivalence relation $\Rightarrow x \sim y$ $\quad$ ($\sim$ is symmetric (ii))

$\Rightarrow x\sim y,\ y \sim z \Rightarrow x \sim z$ $\quad$ ($\sim$ is transitive (iii))

$\Rightarrow z \in [x]$. $\square$

<u>LSIRA 1.5 Prob 5 (Pg 20)</u> Let $\sim$ be a relation on $\mathbb{R}^3$ defined as

$$(x,y,z) \sim (x',y',z') \iff 3x-y+2z = 3x'-y'+2z'.$$

Show that $\sim$ is an equivalence relation. Describe its equivalence classes.

We check that $\sim$ is reflexive, symmetric, and transitive.

<u>Reflexive</u>: $(x,y,z) \sim (x,y,z)$, as $3x-y+2z = 3x-y+2z$. ✓

<u>Symmetric</u>: $(x,y,z) \sim (x',y',z') \Rightarrow (x',y',z') \sim (x,y,z)$ holds as

$$3x-y+2z = 3x'-y'+2z' \Rightarrow$$
$$3x'-y'+2z' = 3x-y+2z. \checkmark$$

$a=b \Rightarrow b=a$ for $a,b \in \mathbb{R}$.

<u>Transitive</u>: $(x,y,z) \sim (x',y',z')$ and $(x',y',z') \sim (x'',y'',z'')$

$\Rightarrow (x,y,z) \sim (x'',y'',z'')$ also holds, as

$$3x-y+2z = 3x'-y'+2z' \text{ and } 3x'-y'+2z' = 3x''-y''+2z''$$
$$\Rightarrow 3x-y+2z = 3x''-y''+2z''. \checkmark$$

$$[(x,y,z)] = \{(x',y',z') \in \mathbb{R}^3 \mid 3x-y+2z = 3x'-y'+2z'\}$$

If we set $3x-y+2z = d \in \mathbb{R}$, then

$$[(x,y,z)] = \{\underline{(x',y',z') \in \mathbb{R}^3 \mid 3x'-y'+2z' = d}\}$$

plane with normal vector $(3,-1,2)$ (or $\begin{bmatrix}3\\-1\\2\end{bmatrix}$) through (x,y,z).

We can describe the equivalence classes as follows.
The equivalence class of a point in $\mathbb{R}^3$ is the plane with normal $(3,-1,2)$ passing through that point.

We write $\mathbb{R}^3/\sim$ for the set of all equivalence classes of $\sim$.

Def If $\sim$ is an equivalence relation on X, then $X/\sim$ is the set of all equivalence classes under $\sim$. "X quotient $\sim$"

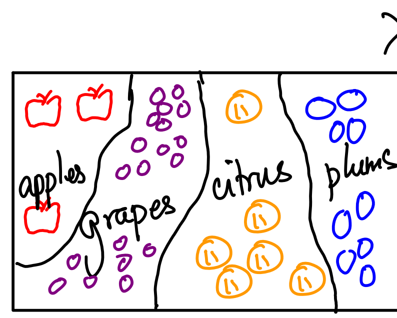
$\mathbb{R}^3/\sim$ here is the set of all planes with normal $(3,-1,2)$.

Note that any point $(x,y,z) \in \mathbb{R}^3$ belongs to exactly one plane with normal $(3,-1,2)$. Also, all such parallel planes together cover all of $\mathbb{R}^3$, i.e., $\mathbb{R}^3/\sim$ is indeed a partition of $\mathbb{R}^3$. Note the similarity to previous example of 45° lines in $\mathbb{R}^2$.

Another example on equivalence classes and Partitions

let X be the set of all fruits in a grocery store. We can group them into fruit types (classes), e.g., apples, citrus, grapes, tomatoes, plums, etc. Note that apples could include honeycrisp, red delicious, etc. (varities of apples)

$\mathcal{P}$: A partition of X into fruit classes may look like this $\longrightarrow$



$$\mathcal{P} = \{\underbrace{\text{apples}}_{P_1}, \underbrace{\text{grapes}}_{P_2}, \underbrace{\text{citrus}}_{P_3}, \underbrace{\text{plums}}_{P_4}, \dots\}$$

Note that any individual fruit belongs to exactly one class. $\mathcal{P}$ is indeed a partition of X.

Equivalence relation $\sim$ on X associated with $\mathcal{P}$

For fruits x, y, $x \sim y$ if x and y are the same fruit type.
$\sim$ is indeed an equivalence relation (can check it's reflexive, symmetric, transitive).

What is the equivalence class $[x]$ of a fruit x?

$[x]$ is the set of all fruits of its type in the store.
e.g., x = Valencia orange, $[x] = \{$set of all citrus fruits$\}$.

What is the quotient space $X/\sim$? $X/\sim$ is the set of all fruit types.

So $X/\sim = \{\text{apples}, \text{citrus}, \dots\}$

Check all problems on equivalence relations from LSIRA.

LSIRA 1.6 Countability

We typically count a set of objects as $1, 2, 3, \ldots$, i.e., by numbering or indexing the first element, then the second one, etc. We can talk about sets being countable (or not) in general.

Def A set A is **countable** if it is possible to list all elements of A as $a_1, a_2, \ldots, a_n, \ldots$

e.g., $\mathbb{N}$ is countable — just list the elements as $1, 2, 3, \ldots$.
set of natural numbers

We could use a little more formal definition of a countable set, than the one given above (as listed in LSIRA).

Def A set A is countable if there exists an injective function $f : A \to \mathbb{N}$.

The function f is the "indexing" or "numbering" function that assigns a separate natural number to each element of A.

Note that finite sets are always countable — we can always list the elements in a sequence. Things are more interesting for infinite sets.

Def If f is also surjective, i.e., it is bijective, then A is **countably infinite**, i.e., it is countable and is infinite.

e.g., $\mathbb{Z}$ is countable.

We can list all integers as

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

This is just one way to list all integers. Other ways could be devised as well.

index: $0 \leftarrow 1$, $1 \leftarrow 2$, $-1 \leftarrow 3$, $2 \leftarrow 4$, $-2 \leftarrow 5$, $3 \leftarrow 6$, $-3 \leftarrow 7$, $\ldots$

Note how the indices are listed. The positive integers are the even entries in the list, and the negative integers (& 0) are the odd entries in the list

Or, we can define $f: \mathbb{Z} \to \mathbb{N}$ as

$$f(z) = \begin{cases} 2z, & z > 0 \\ 1-2z, & z \leq 0 \end{cases}$$

We can specify $f^{-1}(\cdot)$ as follows:

$$f^{-1}(n) = \begin{cases} n/2, & n \text{ even} \\ \frac{-n+1}{2}, & n \text{ odd.} \end{cases}$$

f is bijective, and hence $\mathbb{Z}$ in countably infinite.

Proposition 1.6.1 If A, B are countable, then so is $A \times B$.
↳ Cartesian product

A, B are countable $\Rightarrow \exists$ lists $\{a_n\}, \{b_n\}$ containing all elements of A and B, respectively.

$$\Rightarrow \{\underbrace{(a_1,b_1)}_{1+1}, \underbrace{(a_1,b_2)}_{1+2=3}, \underbrace{(a_2,b_1)}_{2+1=3}, \underbrace{(a_1,b_3)}_{1+3=4}, \underbrace{(a_2,b_2)}_{2+2=4}, \underbrace{(a_3,b_1)}_{3+1=4}, \ldots\}$$

index

is a list containing all elements of $A \times B$.

Note the index trick: we list pairs of elements (a_i, b_j) with $a_i \in \{a_n\}$ and $b_j \in \{b_n\}$ such that the sum of their indices increase as natural numbers. Thus, $i+j=2$, and then all options for $i+j=3$, followed by all options for $i+j=4$, and so on.

This index trick could be used to show other sets are countable, e.g., the cartesian product of k countable sets is countable. ($A_1 \times A_2 \times \cdots \times A_k$, where A_i is countable for $1 \leq i \leq k$).

LSIRA 1.6 Prob 1 (Pg 22) Show that the subset of a countable set is countable.

Let $B \subset A$, where A is countable.

As A is countable, there is a list $a_1, a_2, \dots, a_n, \dots$ such that every $a_i \in A$ is included in the list.

from the list

Let $n_1 \in \mathbb{N}$ be the smallest natural number such that $a_{n_1} \in B$.

And let $n_2 \in \mathbb{N}$, $n_2 > n_1$, be the smallest number such that $a_{n_2} \in B$, and let $n_3 > n_2$, $n_3 \in \mathbb{N}$, be the smallest number such that $a_{n_3} \in B$, and so on.

We form a new list with $b_i = a_{n_i}$, $i = 1, 2, 3, \dots$

$\Rightarrow$ $b_1, b_2, b_3, \dots$ is a listing of all elements in B, ensuring that B is countable. $\square$

indeed, we will miss no elements of B in this process, and all of them are included in the new list.

Check Prop 1.6.2 (in LSIRA): $\bigcup_{n \in \mathbb{N}} A_n$ is countable when A_n is countable $\forall n$.

We can use a similar indexing trick as in Prop. 1.6.1.

Countability is one way to compare two infinite sets. We know $\mathbb{R} \supseteq \mathbb{Q}$, but both have infinitely many elements. Intuitively, we know $\mathbb{R}$ is bigger as it contains irrational numbers in addition to rationals.

We'll first show that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is, in fact, uncountable. More in the next lecture...