MATH401: Lecture 6 (09/04/2025)

Today: * $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable
* ϵ-δ proofs, convergence

Recall: **Proposition 1.6.1** If A, B are countable, then so is $A \times B$.

Proposition 1.6.3 $\mathbb{Q}$ is countable.
$\hookrightarrow$ set of all rational numbers, $\frac{p}{q}$ for $p \in \mathbb{Z}, q \in \mathbb{N}$

This representation includes all negative rationals. Also, $q \in \mathbb{N}$ avoids $q=0$.

We first observe that $\mathbb{Z} \times \mathbb{N}$ is countable, as we showed that $\mathbb{Z}$ and $\mathbb{N}$ are both countable, and then applying Proposition 1.6.1.

$\Rightarrow$ $\mathbb{Z} \times \mathbb{N}$ can be listed as, for instance,
$\{\{(a_1,b_i)\}_{i=1}^{\infty}, \{(a_2,b_i)\}_{i=1}^{\infty}, \dots, \{(a_k,b_i)\}_{i=1}^{\infty}, \dots\}$ where $\{a_n\}$ and $\{b_n\}$ are listings for $\mathbb{Z}$ and $\mathbb{N}$, respectively.

But $\left\{\left\{\frac{a_1}{b_i}\right\}_{i=1}^{\infty}, \left\{\frac{a_2}{b_i}\right\}_{i=1}^{\infty}, \dots, \left\{\frac{a_k}{b_i}\right\}_{i=1}^{\infty}, \dots\right\}$ is a listing of $\mathbb{Q}$. $\square$

Let's consider any rational number, e.g., $\frac{2}{5}$.
How many times does $\frac{2}{5}$ appear in this listing? Once, exactly as $\frac{2}{5}$.

But infinitely many times as a value, because $\frac{2}{5} = \frac{4}{10} = \frac{20}{50} = \dots$

In fact, every rational number appears infinitely many times in this list. But that is not a problem for countability.

We now show that the set of all reals is uncountable.

Theorem 1.6.4 $\mathbb{R}$ is uncountable.

Consider $[0,1] \subset \mathbb{R}$. We show that $[0,1]$ is uncountable.
To get a contradiction, assume that $[0,1]$ is countable.

As there are infinitely many real #s between 0 and 1.
$[0,1]$ is a countably infinite set (under assumption).

We can list all these real numbers as follows:

Note that each number has infinitely many decimal digits (they could be all zeros after some number of places)

$$r_1 = 0.a_{11}\,a_{12}\,a_{13}\cdots$$
$$r_2 = 0.a_{21}\,a_{22}\,a_{23}\cdots$$
$$r_3 = 0.a_{31}\,a_{32}\,a_{33}\cdots$$
$$\vdots \qquad\qquad \vdots$$

a_{ij} = j^{th} decimal digit in the i^{th} real number (in the list).

$a_{ij} \in \{0,1,2,\ldots,9\}$.

We create a new real number in $[0,1]$ as follows.

$$s = 0.d_1 d_2 d_3 \cdots \quad \text{where}$$

$$d_i = \begin{cases} 1 & \text{if } a_{ii} \neq 1, \text{ and} \\ 2 & \text{if } a_{ii} = 1. \end{cases}$$

e.g.,
$$r_1 = 0.02534\ldots$$
$$r_2 = 0.8076\ldots$$
$$r_3 = 0.30914\ldots$$
$$r_4 = 0.00207\ldots$$
$$\vdots$$

Then $s = 0.1211\ldots$

Note that s has infinitely many decimal digits.

So, s is different from r_i for each i.
This contradicts the assumption that $\{r_i\}$ contains **every** real number in $[0,1]$. Hence $[0,1]$ is uncountable.

Since $\mathbb{R} \supset [0,1]$, and $[0,1]$ is uncountable,
$\mathbb{R}$ is also uncountable. $\square$

This is a standard trick we use to show a set is uncountable. We assume it is countable, and start with a listing. Then we identify an element that is distinct from every element in the listing, contradicting the assumption that the listing includes all such elements.

Corollary. The set of irrational numbers is uncountable.

We showed $\mathbb{Q}$ is countable, and $\mathbb{R}$ is uncountable.
The set of irrationals $= \mathbb{R}/\mathbb{Q}$ is hence uncountable.

2.1. ϵ-δ Definitions and Proofs

Norms and Distances

Def The **distance** between $\bar{x} = (x_1, \dots, x_m)$ $\left(\text{or } \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right)$ and $\bar{y} = (y_1, \dots, y_m)$, two points in $\mathbb{R}^m$ is

→ Euclidean distance, by default

$$\|\bar{x} - \bar{y}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_m - y_m)^2}.$$

My notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\theta}$, etc. are vectors → lower case letters with a bar.

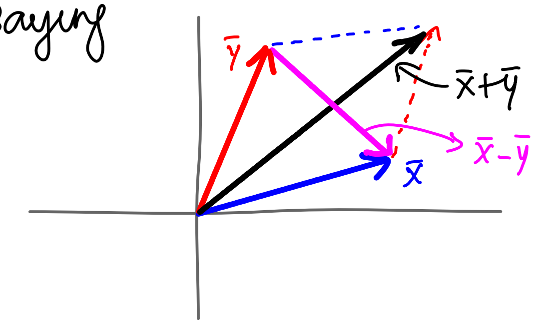
For $m=1$, $\|x-y\| = \sqrt{(x-y)^2} = |x-y|$ → absolute value of $x-y$

think of it as just the distance between two points in $\mathbb{R}$.

Triangle Inequality

$$\forall\, \bar{x}, \bar{y} \in \mathbb{R}^m, \quad \|\bar{x} + \bar{y}\| \le \|\bar{x}\| + \|\bar{y}\|.$$

We could interpret the triangle inequality as saying length of diagonal $\le$ sum of lengths of sides of the parallelogram.

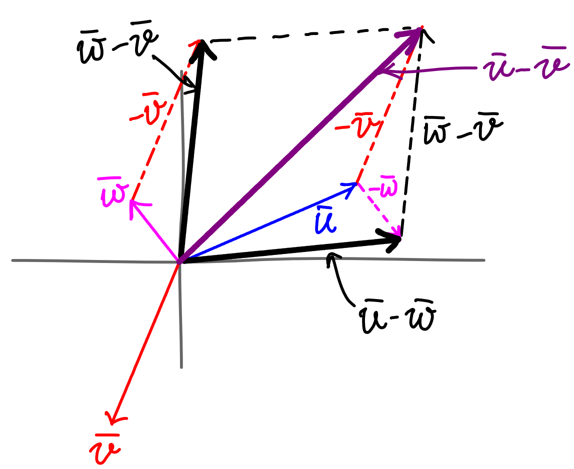


With $\bar{x} = \bar{u} - \bar{w}$, $\bar{y} = \bar{w} - \bar{v}$, we get

$$\|\bar{u} - \bar{v}\| = \|\bar{u} - \bar{w} + \bar{w} - \bar{v}\| \le \|\bar{u} - \bar{w}\| + \|\bar{w} - \bar{v}\|$$

for $\bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^m$

Illustration of the above version in 2D: notice the parallelogram here as well!

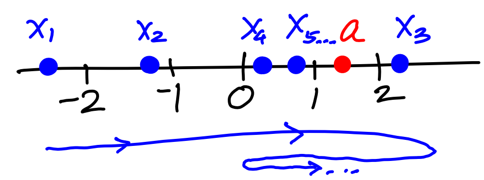


Convergence of Sequences

As a first use of distances, we consider convergence of sequences. How do we say a sequence $\{x_n\}$ converges to a real number a? We should be able to get arbitrarily close to a by going far enough (large n) into the sequence.

Def 2.1.1 A sequence $\{x_n\}$ of real numbers converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$ (no matter how small), there exists an $N \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ for all $n \geqslant N$. We write $\lim\limits_{n\to\infty} x_n = a$.

Here is a pictorial representation of the convergence, with the "path" drawn separately below for clarity.



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Show that if $\{x_n\}$ converges to a, then the sequence $\{Mx_n\}$ converges to Ma. Use the definition of convergence to explain how you choose N.

Given $\{x_n\} \to a$ $\left(\lim\limits_{n\to\infty} x_n = a\right)$ $\Rightarrow$ $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ $\forall n \geqslant N$.

We want to show $\{Mx_n\} \to Ma$. We want to show that $\forall \epsilon > 0$, $\exists N' \in \mathbb{N}$ s.t. $|Mx_n - Ma| < \epsilon$ $\forall n \geqslant N$.

Note that when $M = 0$, the result holds trivially, as $Mx_n = 0$ $\forall n$, and $Ma = 0$. Hence $|Mx_n - Ma| = 0 < \epsilon$ for any $\epsilon > 0$ for $n \geqslant 1$. N'

Also note that both $M > 0$ and $M < 0$ are possible.

Let's assume $M \neq 0$.

First, observe that $|Mx_n - Ma| = |M(x_n - a)| = |M||x_n - a|$.

Note that when $|x_n - a| < \epsilon' = \dfrac{\epsilon}{|M|}$, $|M||x_n - a| < \epsilon$.

But since $\{x_n\} \to a$, given $\epsilon' = \dfrac{\epsilon}{|M|} > 0$, $\exists\ N' \in \mathbb{N}$ s.t. $|x_n - a| < \epsilon'$ for all $n \geq N'$. We can choose $N = N'$, and get

$$|x_n - a| < \epsilon' = \frac{\epsilon}{|M|} \quad \forall n \geq N'$$

$$\Rightarrow |M||x_n - a| = |Mx_n - Ma| < \epsilon \quad \forall\ n \geq N'$$

$\Rightarrow \{Mx_n\}$ converges to Ma. $\square$