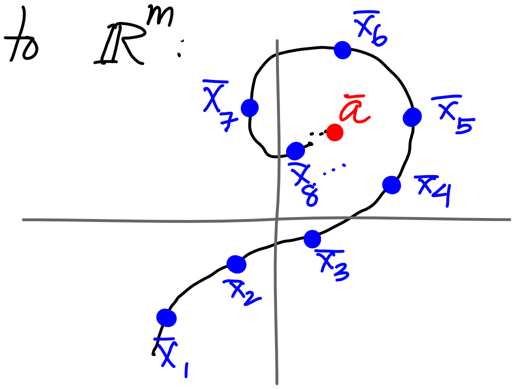
MATH 401: Lecture 7 (09/09/2025)

Today: * convergence in $\mathbb{R}^m$
* continuity of functions

We extend the notion of convergence in $\mathbb{R}$ to $\mathbb{R}^m$:



The definition naturally extends to $\mathbb{R}^m$ once we think of $|x_n - a|$ as the distance between x_n and a.

Def 2.1.2 A sequence $\{\bar{x}_n\}$ of points in $\mathbb{R}^m$ converges to $\bar{a} \in \mathbb{R}^m$ if $\forall \epsilon > 0$, $\exists$ an $N \in \mathbb{N}$ such that $\|\bar{x}_n - \bar{a}\| < \epsilon \;\; \forall\, n \geqslant N$. We write $\lim_{n\to\infty} \bar{x}_n = \bar{a}$.

LSIRA Prob 2.1.3 $\{\bar{x}_n\}$, $\{\bar{y}_n\}$ are two sequences in $\mathbb{R}^m$ where $\{x_n\} \xrightarrow{\text{converges}} \bar{a}$, and $\{\bar{y}_n\} \to \bar{b}$. Then show that $\{\bar{x}_n + \bar{y}_n\}$ converges to $\bar{a} + \bar{b}$.

We want to show: $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that   (any given ϵ)

$$\|(\bar{x}_n + \bar{y}_n) - (\bar{a}_n + \bar{b}_n)\| < \epsilon \quad \forall\, n \geqslant N.$$

same ϵ as our target

Hint, hint, hint! $\|\bar{x} + \bar{y} + \bar{z}\| \leq \|\bar{x}\| + \|\bar{y}\| + \|\bar{z}\|$ by applying triangle inequality twice. We often choose $\epsilon/3$ (instead of $\frac{\epsilon}{2}$) with 3 terms!

We are given $\{\bar{x}_n\} \to \bar{a}$, $\{\bar{y}_n\} \to \bar{b}$, so

$\exists N_1 \in \mathbb{N}$ s.t. $\|\bar{x}_n - \bar{a}\| < \frac{\epsilon}{2} \;\; \forall\, n \geqslant N_1$ and

$\exists N_2 \in \mathbb{N}$ s.t. $\|\bar{y}_n - \bar{b}\| < \frac{\epsilon}{2} \;\; \forall\, n \geqslant N_2$.

$\Rightarrow$ for $N = \max\{N_1, N_2\}$, we get

$$\begin{aligned}\|(\bar{x}_n + \bar{y}_n) - (\bar{a} + \bar{b})\| &= \|(\bar{x}_n - \bar{a}) + (\bar{y}_n - \bar{b})\| \\ &\leq \|\bar{x}_n - \bar{a}\| + \|\bar{y}_n - \bar{b}\| \quad \text{by triangle inequality} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{as } N \geqslant N_1,\ N \geqslant N_2.\end{aligned}$$

$\Rightarrow \{\bar{x}_n + \bar{y}_n\} \to \bar{a} + \bar{b}$. $\square$

Continuity

$f: \mathbb{R} \to \mathbb{R}$. When is f continuous at $x=a$?

For sequences $\{x_n\} \to a$, we go "far enough out", i.e., $\forall n \geq N \in \mathbb{N}$. Instead of $N \in \mathbb{N}$, here we say $\exists\, \delta > 0$ such that if $|x-a| < \delta$ then $|f(x)-f(a)| < \epsilon$ (for any given $\epsilon > 0$). In other words, $f(x)$ gets close enough to $f(a)$ whenever x is close enough to a!

Def 2.1.4 The function $f: \mathbb{R} \to \mathbb{R}$ is **continuous** at $a \in \mathbb{R}$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $|f(x)-f(a)| < \epsilon$ whenever $|x-a| < \delta$.

Equivalently, if $|x-a| < \delta$ then $|f(x)-f(a)| < \epsilon$.

We naturally extend the definition to $\mathbb{R}^m$ using distances/norms.

LSIRA uses $\mathbf{F}$ (bold upper case F)

Def 2.1.7 The function $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\bar{a} \in \mathbb{R}^n$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $\|\bar{f}(\bar{x}) - \bar{f}(\bar{a})\| < \epsilon$ whenever $\|\bar{x} - \bar{a}\| < \delta$.

By restricting our attention to a subset A of $\mathbb{R}^n$, we naturally extend the above definition to subsets of interest.

Def 2.1.8 Let $A \subset \mathbb{R}^n$, and $\bar{a} \in A$.
The function $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\bar{a} \in A$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $\|\bar{f}(\bar{x}) - \bar{f}(\bar{a})\| < \epsilon$ whenever $\|\bar{x} - \bar{a}\| < \delta$ and $\bar{x} \in A$.

LSIRA Section 2.1 Prob 4 (extension): If $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,3$ are all continuous at $a \in \mathbb{R}$, the show that so is $f_1 + f_2 - f_3$. (i.e., show $f_1(x) + f_2(x) - f_3(x)$ is continuous at $x = a$).

Prob 4 considers $f+g$ for two functions f, g.

Let $g(x) = f_1(x) + f_2(x) - f_3(x)$. We want to show that

$\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|g(x) - g(a)| < \epsilon$ whenever $|x - a| < \delta$.

We know: since $f_i(x)$ are continuous at $x = a$,

$\exists \delta_i > 0$ s.t. $|f_i(x) - f_i(a)| < \frac{\epsilon}{3}$ whenever $|x - a| < \delta_i$, $i = 1,2,3$.

Let $\delta = \min\limits_{i=1,2,3} \{\delta_i\}$. Then

We want x to as close to a as required in each case!

e.g., if $\delta_1 = 0.1$, $\delta_2 = 0.05$ and $\delta_3 = 0.08$, then $\delta \le 0.05$ works!

$$
\begin{aligned}
|g(x) - g(a)| &= \left| \left(f_1(x) + f_2(x) - f_3(x)\right) - \left(f_1(a) + f_2(a) - f_3(a)\right) \right| \\
&= \left| \left(f_1(x) - f_1(a)\right) + \left(f_2(x) - f_2(a)\right) + \left(f_3(a) - f_3(x)\right) \right| \\
&\le |f_1(x) - f_1(a)| + |f_2(x) - f_2(a)| + \underbrace{|f_3(a) - f_3(x)|}_{= |f_3(x) - f_3(a)|} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \text{as } \delta \le \delta_i \text{ for } i = 1,2,3 \\
&= \epsilon \quad \text{whenever } |x - a| < \delta.
\end{aligned}
$$

by triangle inequality (applied twice)

$\square$

LSIRA Proposition 2.1.9 Let $g:\mathbb{R}\to\mathbb{R}$ be continuous at $a\in\mathbb{R}$, and $g(a)\neq 0$.

Show that $h(x)=\dfrac{1}{g(x)}$ is continuous at $x=a$.

Need to show: $\forall\,\epsilon>0$, $\exists\,\delta>0$ s.t. $|h(x)-h(a)|<\epsilon$ whenever $|x-a|<\delta$.

We want to show that

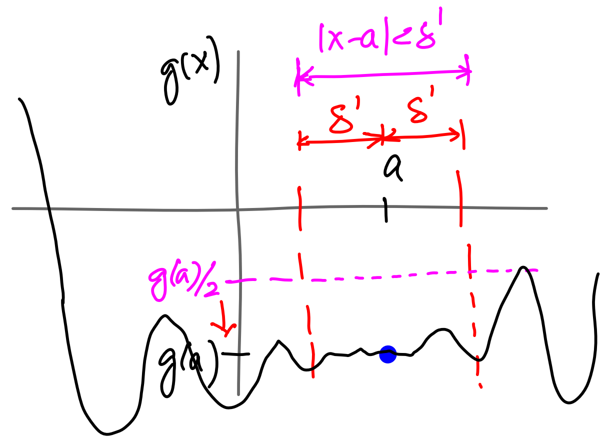
$$|h(x)-h(a)| = \left|\frac{1}{g(x)}-\frac{1}{g(a)}\right| < \epsilon.$$

$$\left|\frac{1}{g(x)}-\frac{1}{g(a)}\right| = \left|\frac{g(a)-g(x)}{g(x)\,g(a)}\right| = \frac{|g(x)-g(a)|}{|g(x)|\,|g(a)|}$$

$\neq 0$

We want to show that $|g(x)|$ is not too small. Else, the fraction could be too large.

There must exist some $\delta'>0$ such that $|g(x)|>\dfrac{|g(a)|}{2}$ whenever $|x-a|<\delta'$, as $g(a)\neq 0$.



In the picture here, notice that $g(x)$ lies "below" the $\frac{g(a)}{2}$ level, i.e., far enough away from zero, when $|x-a|<\delta'$.

Also, $g(x)$ is continuous at $x=a \implies$

$\exists\,\delta''>0$ s.t. $|g(x)-g(a)|<\epsilon'$ whenever $|x-a|<\delta''$.

Pick $\delta = \min\{\delta', \delta''\}$. Then we get that

$$\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| = \frac{|g(x)-g(a)|}{|g(x)|\,|g(a)|} < \frac{\epsilon'}{\frac{|g(a)|}{2}\,|g(a)|} = \frac{2\epsilon'}{|g(a)|^2}$$

whenever $|x-a| < \delta$.

If we choose $\epsilon' = \frac{|g(a)|^2}{2}\epsilon$, so that $\frac{2\epsilon'}{|g(a)|^2} = \epsilon$,

we get that $\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| < \epsilon$ whenever $|x-a| < \delta$.

Hence $\frac{1}{g(x)}$ is continuous at $x = a$

$\square$

In the next section, we consider the setting where the target or candidate limit (a) is not given to us.
Can we still conclude that $\{\bar{x}_n\}$ converges? When?