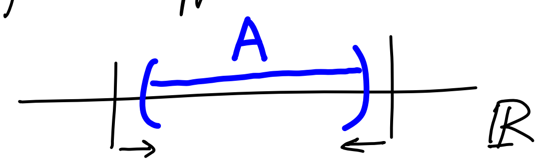
MATH 401: Lecture 8 (09/11/2025)

Today: * completeness
* sup, inf, lim sup, lim inf

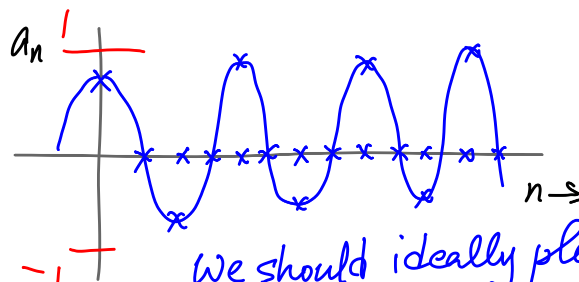
Completeness (LSIRA 2.3)

If we don't know the limit target $\bar{a}$, can we still say $\{\bar{a}_n\}$ converges?
If $\{\bar{a}_n\}$ "behaves nicely" and $\bar{a}_n$'s are in a "nice space", then yes!

Here is an intuition for what we mean by "nice space". Suppose $a_n \in A$ where A is a "finite" interval (open or closed). Then we can be sure that the a_n's cannot become arbitrarily large or arbitrarily small.

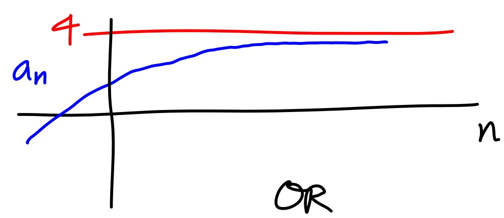


But in this example, the a_n's belong to a bounded interval $[-1,1]$, but they are not "behaving nicely" as the values oscillate between 1 and -1.



We should ideally plot the discrete pts (x), instead of the continuous curve

But if the a_n's are increasing and are bounded from above, or decreasing and bounded from below, we can conclude that $\{a_n\}$ converges!



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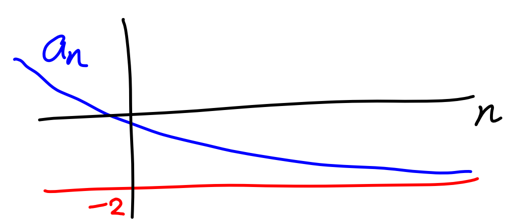
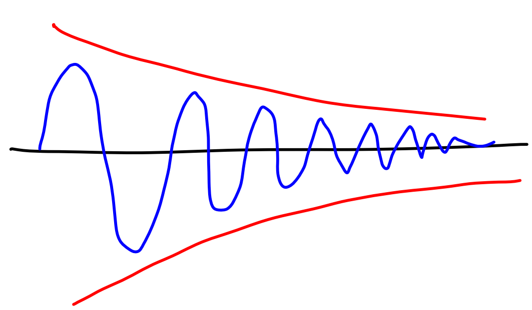


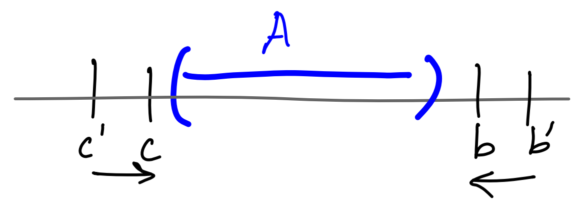
Finally, even if a_n's are oscillating, and hence not increasing/decreasing, it could still be nice if the oscillations become smaller and smaller — as shown here. Intuitively, we want the upper and lower "envelopes" to get closer and closer.



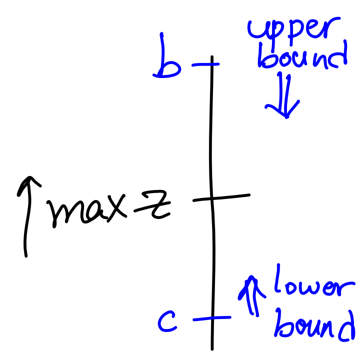
We formalize these intuitive notions of "nice" space and "nice" behavior.

Def A nonempty set $A \subset \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that $a \leq b \;\forall a \in A$, and is **bounded below** if there exists $c \in \mathbb{R}$ such that $a \geq c \;\forall a \in A$. We refer to b as an **upper bound**, and c as a **lower bound**.

If b is an upper bound, then any $b' > b$ is also an upper bound. Similarly, and $c' < c$ is also a lower bound.



We usually want to find a smallest upper bound, and a largest lower bound. This idea is ubiquitous in optimization, where finding the correct maximum value for a function $z = f(\bar{x})$ may be hard, but it may be easier to obtain lower/upper bounds. In order to get as best a handle on the actual max z value, we try to find the smallest upper bound, and the biggest lower bound that work.



In the same way, we want to "estimate" A as accurately as possible by finding the smallest upper bound and the largest lower bound for the set.

The Completeness Principle

Every nonempty subset A of $\mathbb{R}$ that is bounded above has a least upper bound. This bound is called the **supremum of A**, written $\sup A$.

Similarly every nonempty subset A of $\mathbb{R}$ that is bounded below has a greatest lower bound, called the **infimum of A**, written $\inf A$.

LSIRA 2.2 Problem 1 Argue that $\sup\, [0,1) = 1$ and $\sup\, [0,1] = 1$.

Let $A = [0,1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$. So $x \in A$ can be arbitrarily close to 1, i.e., $x = 1-\epsilon$, $\epsilon > 0$, arbitrarily small. Hence any $1-\epsilon$ cannot be an upper bound for A, since $\forall \epsilon > 0$, $\exists\ 1-\epsilon' \in A$ s.t. $1-\epsilon' > 1-\epsilon$.

$\Rightarrow$ $b \geq 1$ satisfies $x \leq b\ \forall x \in A$, and hence $\sup A = 1$.

The same argument holds for $[0,1]$ too. Note that the sup is in A in the latter case, but $\sup A \notin A$ for $A = [0,1)$.

So, what is the big deal about the completeness principle?

First, it does not hold over $\mathbb{Q}$ (rationals), as, e.g.,

$$A = \{x \in \mathbb{R} \mid x^2 < 3\} \text{ has } \sup A = \sqrt{3}. \text{ But}$$

$$B = \{x \in \mathbb{Q} \mid x^2 < 3\} \text{ has no supremum in } \mathbb{Q}!$$

$\rightarrow$ $\sqrt{3}$ is irrational, and we can get arbitrarily close to $\sqrt{3}$ using rational numbers!

We say that $\mathbb{Q}$ does not satisfy completeness principle.

Monotone Sequences, lim sup, lim inf

We now describe sequences that behave "nicely" like the bounded sets introduced earlier. We then consider how to handle sequences that are not as "nice".

"nondecreasing" if you want $a_{n+1} > a_n$ to mean "increasing"

Def A sequence $\{a_n\}$ in $\mathbb{R}$ is **increasing** if $a_{n+1} \geq a_n \; \forall n$.

A sequence $\{a_n\}$ in $\mathbb{R}$ is **decreasing** if $a_{n+1} \leq a_n \; \forall n$.

$\{a_n\}$ is **monotone** if it is either increasing or decreasing.

$\{a_n\}$ is **bounded** if $\exists \, M \in \mathbb{R}$ s.t. $|a_n| \leq M \; \forall n$.

LSIRA Theorem 2.2.2 Every monotone bounded sequence in $\mathbb{R}$ converges to a number in $\mathbb{R}$.

we do not specify which number!

Proof (for increasing sequences). We proceed in two steps.

1. $\{a_n\}$ is bounded $\Rightarrow$ $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ is bounded. set

   $\Rightarrow \exists \, a \in \mathbb{R}$ such that $\sup A = a$. $\rightarrow$ using completeness of $\mathbb{R}$

2. a is the least upper bound. $\rightarrow$ We show $\{a_n\} \rightarrow a$

   $\Rightarrow a - \epsilon$ is not an upper bound for any $\epsilon > 0$.

   $\{a_n\}$ is increasing $\Rightarrow$ $a - \epsilon < a_n \leq a \;\; \forall n \geq N$ for some N.

   $\Rightarrow |a - a_n| < \epsilon \;\; \forall \, n \geq N$, i.e., $\{a_n\}$ converges.

   $a_n - a > -\epsilon$ and $a - a_n < \epsilon$

$\square$

But what if $\{a_n\}$ is not monotone and/or not bounded? Can we still say something about $\{a_n\}$ as $n \to \infty$?

Given a general sequence $\{a_n\}$, we define two related sequences that are monotone themselves.

Def Given $\{a_k\}$, $a_k \in \mathbb{R}$, we define two new sequences $\{M_n\}$ and $\{m_n\}$ as follows.

$$M_n = \sup \{a_k \mid k \geq n\} \quad \text{and}$$

$$m_n = \inf \{a_k \mid k \geq n\}.$$

$M_n = \infty$, $m_n = -\infty$ are allowed here.

M_n "captures" how large $\{a_k\}$ can be "after" n, and m_n captures how small $\{a_k\}$ can be "after" n.

Note that $\{M_n\}$ and $\{m_n\}$ are monotone!

$\{M_n\}$ is decreasing, as suprema are taken over smaller subsets.
and $\{m_n\}$ is increasing, as infima are taken over smaller subsets.

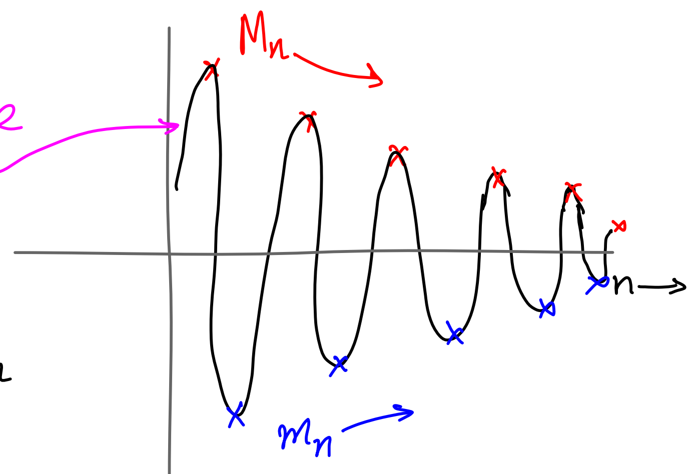
e.g., consider $A = \{1, 2, \ldots, 10\}$. The largest number in A cannot be bigger than the largest number in $A' = \{1, 2, \ldots, 7\}$, or in any $A' \subset A$, in general.

$$\Rightarrow \lim_{n\to\infty} M_n \text{ and } \lim_{n\to\infty} m_n \text{ exist!}$$

Def The **limit superior** or lim sup of the original sequence $\{a_n\}$ is $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} M_n$.

The **limit inferior** of $\{a_n\}$ is $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n$.

We ideally want to draw a sequence of points "......" in place of the continuous curve here



It appears while $\{x_n\}$ may be "oscillating" the upper bounds M_n and lower bounds m_n appear to be converging. Hence, $\{a_n\}$ also appears to converge! But we could have $\{a_n\}$ oscillate forever, even when M_n and m_n are finite $\forall n \geq N$.

LSIRA 2.2 Problem 4

Let $a_n = (-1)^n$. What is $\limsup_{n\to\infty} a_n$? $\liminf_{n\to\infty} a_n = ?$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} M_n = 1.$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n = -1.$$

Note than $a_n = 1 \ \forall\ n = 2k$, and $a_n = -1 \ \forall\ n = 2l+1$.

Hence $a_n \leq 1 \ \forall\ n$, and $a_n \geq -1 \ \forall\ n$.

In fact, $\{M_n\}$ and $\{m_n\}$ behave identical to $\{a_n\}$ here!

In the above problem, even though lim sup and lim inf are both finite, they are not equal, and we cannot say anything about $\{a_n\}$ converging to a limit. But when the lim sup and lim inf are equal, we get the picture drawn earlier, with $\{a_n\}$ converging to that value!

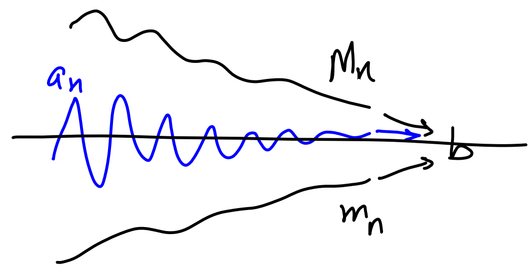
LSIRA Proposition 2.2.3 Let $\{a_n\}$ be a sequence of real numbers.

Then $\lim\limits_{n\to\infty} a_n = b$ if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b.$$

b can $\pm\infty$ here!

($\Leftarrow$) Assume $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$



$$\Rightarrow \lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = b$$

Also, $m_n \le a_n \le M_n \quad \forall n$

$$\Rightarrow \lim_{n\to\infty} a_n = b.$$

(by "squeeze law" or "squeeze theorem"; LSIRA 2.2 Problem 2 — assigned in HW4!)

We'll finish the proof in the next lecture..