MATH 401: Lecture 9 (09/16/2025)

Today: * Cauchy sequences
* Intermediate value theorem (IVT)

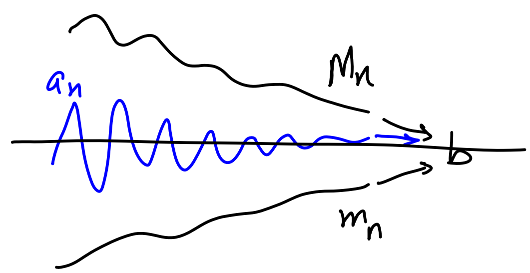
We first present the proof of Proposition 2.2.3...

LSIRA Proposition 2.2.3 Let $\{a_n\}$ be a sequence of real numbers. Then $\lim\limits_{n\to\infty} a_n = b$ iff $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$.

b can be $\pm\infty$ here!

($\Leftarrow$) Assume $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$

$$\Rightarrow \lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = b$$



Also, $m_n \le a_n \le M_n \ \forall n$

$$\Rightarrow \lim_{n\to\infty} a_n = b.$$

(by "squeeze law" or "squeeze theorem"; LSIRA 2.2 Problem 2 — assigned in Hw4!)

($\Rightarrow$) Assume $\lim\limits_{n\to\infty} a_n = b$, and $b \in \mathbb{R}$.

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - b| < \epsilon \quad \forall n \geqslant N$.

$\Leftrightarrow$ $a_n - b < \epsilon$ and $b - a_n < \epsilon$

$|x| < 5$
$\Rightarrow$ $-x < 5$ and $x < 5$

$\Rightarrow b - \epsilon < a_n < b + \epsilon \quad \forall n \geqslant N$

$\Rightarrow b - \epsilon < m_n < b + \epsilon$ and
$b - \epsilon < M_n < b + \epsilon \quad \forall n \geqslant N$

We will repeatedly use this trick of splitting $|x-y| < \epsilon$ into $x - y < \epsilon$ and $y - x < \epsilon$

Since the result holds for any $\epsilon > 0$,

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b.$$

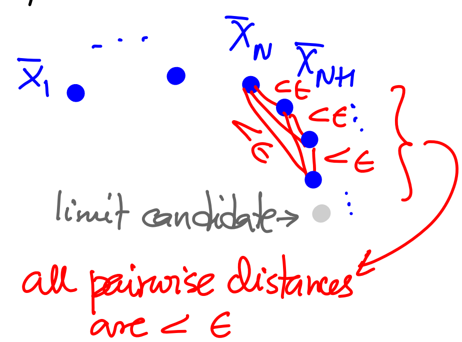
$\square$

Cauchy Sequences

We extend the idea of completeness in $\mathbb{R}$ to $\mathbb{R}^m$. But there is no natural way to order points in $\mathbb{R}^m$ (as in $\mathbb{R}$). Instead, we say the points get closer and closer to each other.

Def 2.2.4 A sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$ is called a **Cauchy sequence** if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\|\bar{x}_n - \bar{x}_k\| < \epsilon \;\; \forall n, k \geq N$.

n, k are two indices, and represent any two points that are both far out enough into the sequence $(n, k \geq N)$



Completeness Result in $\mathbb{R}^m$

Theorem 2.2.5 The sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$ converges **iff** it is cauchy.

This is an iff result. We prove both directions, but one of them is easier than the other. We show the easy direction in $\mathbb{R}^m$, but the reverse direction in $\mathbb{R}$ (and can be extended to $\mathbb{R}^m$).

Proposition 2.2.6 All convergent sequences in $\mathbb{R}^m$ are cauchy.

Proof Let $\{\bar{a}_n\}$ converge to $\bar{a}$ in $\mathbb{R}^m$.

We want to show $\|\bar{a}_n - \bar{a}_k\| < \epsilon \;\; \forall n, k \geq N$ for some $N \in \mathbb{N}$.

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|\bar{a}_n - \bar{a}\| < \frac{\epsilon}{2}, \forall n \geq N$.

Ideally, we use ϵ' here, and then choose $\epsilon' = \frac{\epsilon}{2}$.

$\Rightarrow$ If $n, k \geq N$, then

$$\|\bar{a}_n - \bar{a}_k\| = \|\bar{a}_n - \bar{a} + \bar{a} - \bar{a}_k\| \leq$$

triangle inequality

$$\|\bar{a}_n - \bar{a}\| + \|\bar{a} - \bar{a}_k\| < \underbrace{\frac{\epsilon}{2} + \frac{\epsilon}{2}}_{} = \epsilon \quad \forall n \geq N.$$

now we see why we chose $\frac{\epsilon}{2}$!

$\Rightarrow \{\bar{a}_n\}$ is cauchy. $\square$

We present proof for the reverse direction in $\mathbb{R}$. We can repeat this argument for each dimension to prove the result in $\mathbb{R}^m$. We need a lemma first.

Lemma 2.2.7 Every Cauchy sequence $\{a_n\}$ in $\mathbb{R}$ is bounded.

Want to show: $|a_n| \leq M$ for some $M \in \mathbb{R}$. note, $M \geq 0$

$\{a_n\}$ is cauchy $\Longrightarrow |a_n - a_k| < \epsilon \;\; \forall\, n,k \geq N \in \mathbb{N}$ for any $\epsilon > 0$.

$\Longrightarrow |a_n - a_N| < 1$ (for $\epsilon = 1$) the definition applies for any ϵ, so we choose $\epsilon = 1$. After all, we just need to find a valid bound

$\Longrightarrow a_n - a_N < 1$ and $a_N - a_n < 1$

$\Longrightarrow a_n < a_N + 1$ and $a_n > a_N - 1 \;\; \forall\, n \geq N$.

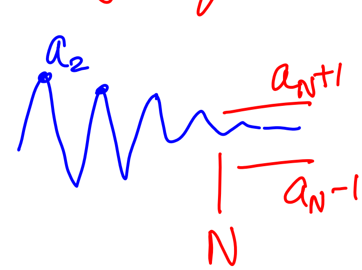
$\Longrightarrow M = \max\{a_1, a_2, \dots, a_{N-1}, a_N + 1\}$ is an upper bound, and

$m = \min\{a_1, a_2, \dots, a_{N-1}, a_N - 1\}$ is a lower bound. $\square$

Could also get $|a_n| - |a_N| \leq |a_n - a_N| < 1$

$\Longrightarrow |a_n| \leq |a_N| + 1$.

We could have a larger number among $a_1, a_2, \dots, a_{N-1}$, which are not considered earlier since the Cauchy definition stipulates $n, k \geq N$.



Proposition 2.2.8 All Cauchy sequences in $\mathbb{R}$ converge.

Proof $\{a_n\}$ is cauchy $\Rightarrow$ $\{a_n\}$ is bounded (by Lemma 2.2.7).

$\Rightarrow M = \limsup\limits_{n\to\infty} a_n$ and $m = \liminf\limits_{n\to\infty} a_n$ are both finite.

We can use Proposition 2.2.3 now, if we can show $M = m$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - a_k| < \epsilon \quad \forall n, k \geq N$.

In particular, $|a_n - a_N| < \epsilon \quad \forall n \geq N$. (taking $k = N$)

$\Rightarrow \quad a_n - a_N < \epsilon \quad$ and $\quad a_N - a_n < \epsilon \quad \forall n \geq N$

$\Rightarrow \quad a_n < a_N + \epsilon \quad$ and $\quad a_n > a_N - \epsilon$

i.e., $a_N - \epsilon < a_n < a_N + \epsilon \quad \forall n \geq N$ holds for any $\epsilon > 0$.

$\Rightarrow \quad M_n = \sup\{a_k \mid k \geq n\} < a_N + \epsilon$ ADD

$-\left(m_n = \inf\{a_k \mid k \geq n\} > a_N - \epsilon\right) \Rightarrow \quad -m_n < -a_N + \epsilon$

$\Rightarrow \quad M_n - m_n < 2\epsilon \quad \forall n \geq N$ and for any $\epsilon > 0$, arbitrary.

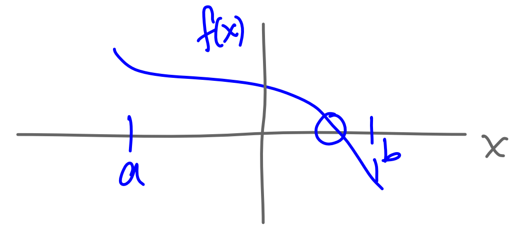
$\Rightarrow \quad M = m$ (as $n \to \infty$).

$\square$

We now present four fundamental theorems, the proofs of which use many of the results we have presented. These theorems are quite fundamental in analysis, and also finds use in many applied domains as well.

Intermediate Value Theorem

This is a rather straightforward result to understand — if a function goes from above the x-axis to below it, and is continuous, then it must cross the x-axis.



Theorem 2.3.1 (Intermediate Value Theorem) Assume $f:[a,b]\to\mathbb{R}$ is continuous, and $f(a)$ and $f(b)$ have opposite signs. Then there exists $c\in(a,b)$ such that $f(c)=0$.

We will use a characterization of continuity using sequences in the proof (from LSIRA 2.1, actually!).

Proposition 2.1.5 $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x=a$ iff $\lim_{n\to\infty} f(x_n)=f(a)$ for all sequences $\{x_n\}$ that converge to a.

Proof

($\Rightarrow$) Assume f is continuous at $x=a$.

Consider $\{x_n\}\to a$, i.e., $\lim_{n\to\infty} x_n = a$.

Need to show: $\forall \epsilon>0, \exists N\in\mathbb{N}$ s.t. $|f(x_n)-f(a)|<\epsilon$ $\forall n\geq N$.

$\Rightarrow$ $\exists\, \delta>0$ s.t. $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$. same

$\exists\, N'\in\mathbb{N}$ s.t. $|x_n-a|<\delta$ whenever $n\geq N'$. $\longrightarrow$ plays the "role of ϵ," i.e. the convergence definition must hold for any $\epsilon>0$, and here we choose $\epsilon=\delta$.

$\Rightarrow$ If $n\geq N'$, then $|f(x_n)-f(a)|<\epsilon$, as $|x_n-a|<\delta$.

$\Rightarrow$ $\{f(x_n)\}\longrightarrow f(a)$. Reverse direction in the next lecture...