MATH 464 – Lecture 10 (02/09/2023)

Today: * correspondence between bfs's and vertices
* degenerate bfs.

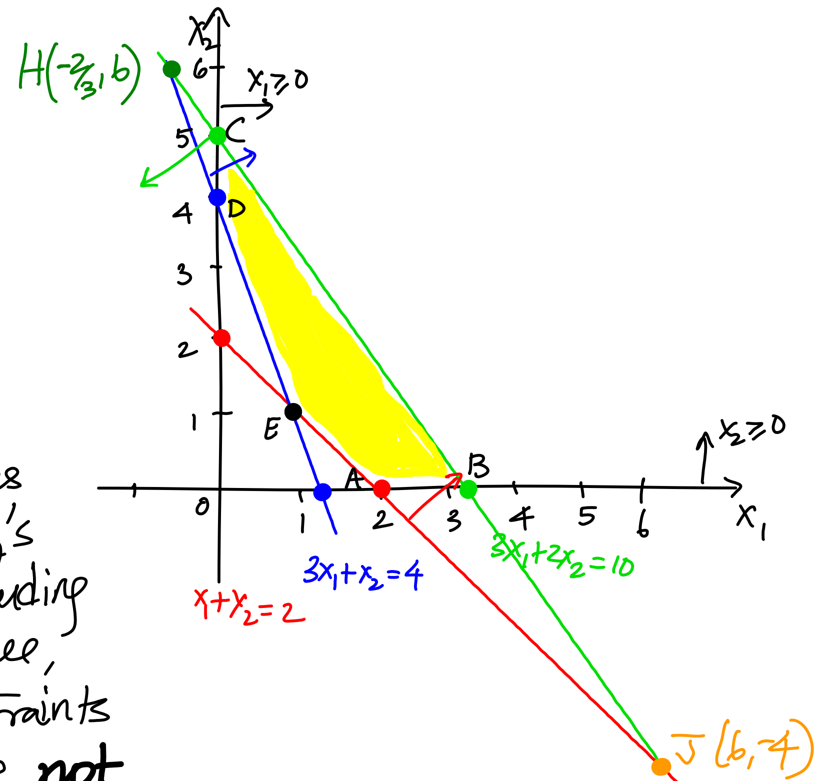
Recall: Procedure to construct basic solutions and bfs's

With $A=[B\ N]$, $\bar{x}=\begin{bmatrix}\bar{x}_B\\ \bar{x}_N\end{bmatrix}$, $A\bar{x}=\bar{b} \Rightarrow B\bar{x}_B+N\bar{x}_N=\bar{b}$

We set $\bar{x}_N=\bar{0} \Rightarrow B\bar{x}_B=\bar{b} \Rightarrow \bar{x}_B=B^{-1}\bar{b}$.

$$\left.\begin{aligned} x_1+x_2-x_3 \quad\quad &= 2\\ 3x_1+x_2 \quad -x_4 \quad &= 4\\ 3x_1+2x_2 \quad\quad +x_5 &= 10 \end{aligned}\right\}$$
$x_j \geq 0\ \forall j$

$$A=\begin{bmatrix}1&1&-1&0&0\\3&1&0&-1&0\\3&2&0&0&1\end{bmatrix}_{3\times 5} \quad \bar{b}=\begin{bmatrix}2\\4\\10\end{bmatrix}$$



We could identify the basis indices at any vertex by identifying the x_j's that are >0 at that vertex, including slack/excess variables. For instance, at $A(2,0)$, $x_1>0$. Also, the constraints $3x_1+x_2 \geq 4$ and $3x_1+2x_2 \leq 10$ are **not** active at A, and hence $x_4>0$ and $x_5>0$. So $B_{ind}=[B(1)\ B(2)\ B(3)]=[1\ 4\ 5]$ at the bfs $\equiv$ A.

We can use the same method to identify basic solutions for $H(-\frac{2}{3},6)$ and $J(6,-4)$. At H, x_1, x_2, and x_3 are $\neq 0$ (not all of them are >0 here!). Hence $B_{ind}=[1\ 2\ 3]$ at H.

as they're not feasible!

Similarly, $B_{ind}=[1\ 2\ 4]$ at J.

See MATLAB session for details.

We can find the bfs in the standard form LP corresponding to each corner point by inspection. In other words, we do not necessarily have to enumerate all $\binom{n}{m}$ choices for bases.

For instance, at the vertex $E(1,1)$, the constraints $x_1+x_2 \geq 2$ and $3x_1+x_2 \geq 4$ are active, while $3x_1+2x_2 \leq 10$ is not binding. Hence $x_3 = x_4 = 0$ and $x_5 > 0$ in the corresponding bfs from the standard form LP. Also notice that both x_1 and x_2 are > 0. Hence, $\{x_1, x_2, x_5\}$ is indeed the corresponding basis.

Similarly, at $C(0,5)$, we see that both $x_1+x_2 \geq 2$ and $3x_1+x_2 \geq 4$ are not active, while $3x_1+2x_2 \leq 10$ and $x_1 \geq 0$ are binding. Hence $x_3 > 0$ and $x_4 > 0$, but $x_5 = 0$. Also, $x_2 = 5$ is > 0. Thus $\{2,3,4\}$ is the set of basic columns.

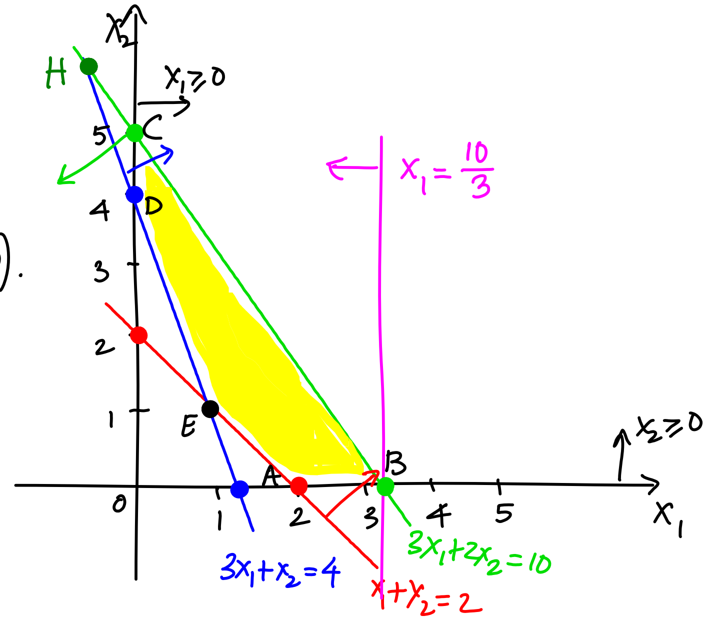
The following table summarizes the correspondences between the bfs's in the standard form polyhedron and the vertices identified in the 2D picture.

det(B) ≠ 0 confirms that the basis matrix is invertible in each instance.

Vertex	B(1)	B(2)	B(3)	det(B)	x_1	x_2	x_3	x_4	x_5
					BFS				
A	1	4	5	−1	2	0	0	2	4
B	1	3	4	3	10/3	0	4/3	6	0
C	2	3	4	2	0	5	3	1	0
D	2	3	5	1	0	4	2	0	2
E	1	2	5	−2	1	1	0	0	5

Degeneracy

Recall adding $x_1 \le \frac{10}{3}$ to the original LP:



Now, the three constraints $3x_1+2x_2 \le 10$, $x_1 \le \frac{10}{3}$ and $x_2 \geq 0$ are active at $B(\frac{10}{3}, 0)$.

We need only two LI constraints to define a BFS here. Hence, B is a degenerate bfs.

In the standard form, we would have $x_1 + x_6 = \frac{10}{3}$, where x_6 is the slack variable for the constraint $x_1 \le \frac{10}{3}$. Notice that we get $x_2 = x_5 = x_6 = 0$ now. We have $n=6$ and $m=4$, and hence more than $n-m=2$ x_j's are zero. This is an illustration of the condition for a bfs to be degenerate in general.

Def (Degeneracy in standard form) A basic solution $\bar{x}$ of P in standard form is **degenerate** if more than $n-m$ x_j's are zero.

But we could have identified B as a degenerate vertex from the 2D picture itself, from the fact that more than $n(=2)$ LI constraints (3 here) are active at B. Indeed, this condition holds more generally as well.

Def A basic solution $\bar{x}^* \in \mathbb{R}^n$ of P (in general form) is a **degenerate basic solution** if more than n constraints are active at $\bar{x}^*$.

A degenerate basic solution that is feasible is called a **degenerate bfs**.

* In 2D, a basic solution has 3 or more lines corresponding to constraints satisfied as equations meeting at the point.
  ↳ (including $x_j \geq 0$)

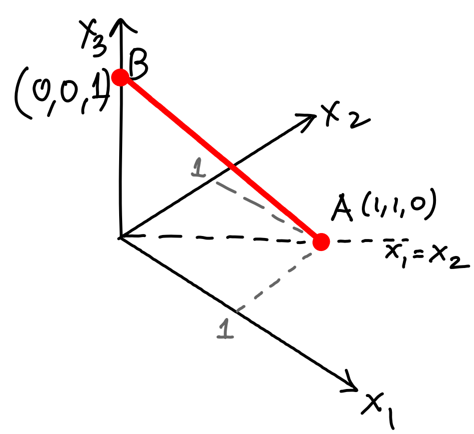
* In $\mathbb{R}^n$, we need $n+1$ or more constraints active at a degenerate basic solution. Hence, if $P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \geq \bar{b}\}$, $A_{m\times n}$, $m \leq n$, there are no degenerate basic solutions.

Degeneracy may not be much of a problem in small instances. But when we are solving large instances of LP, it could create inefficiencies. The typical algorithm tries to move from one bfs to an adjacent bfs such that the objective function value improves, or at least does not become worse. It could happen that we cycle through several degenerate bfs's before ultimately jumping to a vertex that actually strictly improves the objective function value.

But as we are going to see, degeneracy depends very much on how we represent the polyhedron. Indeed, we could throw out $x_1 \leq 1/3$ without changing the polyhedron, and remove the degeneracy at B.

Here is another example.

Consider the polyhedron $P = \{\bar{x} \in \mathbb{R}^3 \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2;\ x_j \geq 0, j = 1,2,3\}$ with $m = 2$, $n = 3$.



A (1,1,0) is a non-degenerate vertex, but B(0,0,1) is a degenerate bfs.

$x_1 - x_2 = 0$, $x_1 + x_2 + 2x_3 = 2$, and $x_3 \geq 0$ are active at $A(1,1,0)$.
Indeed, these three constraints are LI. The corresponding vectors of the constraints are $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

$$x_1 - x_2 = 0 \equiv \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0; \quad x_1 + x_2 + 2x_3 = 2 \equiv \begin{bmatrix}1 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 2.$$

At $B(0,0,1)$, $x_1 - x_2 = 0$, $x_1 + x_2 + 2x_3 = 2$, $x_1 \geq 0$, and $x_2 \geq 0$ are active.
Of course, the four constraints in 3D are not LI.

Alternatively, we can describe the same polyhedron as

$$P = \{\bar{x} \in \mathbb{R}^3 \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2;\ x_1 \geq 0,\ x_3 \geq 0\}.$$ no $x_2 \geq 0$.

For this description, $B(0,0,1)$ is NOT degenerate, as now we have only 3 active constraints at B, namely, $x_1 - x_2 = 0$, $x_1 + x_2 + 2x_3 = 2$, and $x_1 \geq 0$.

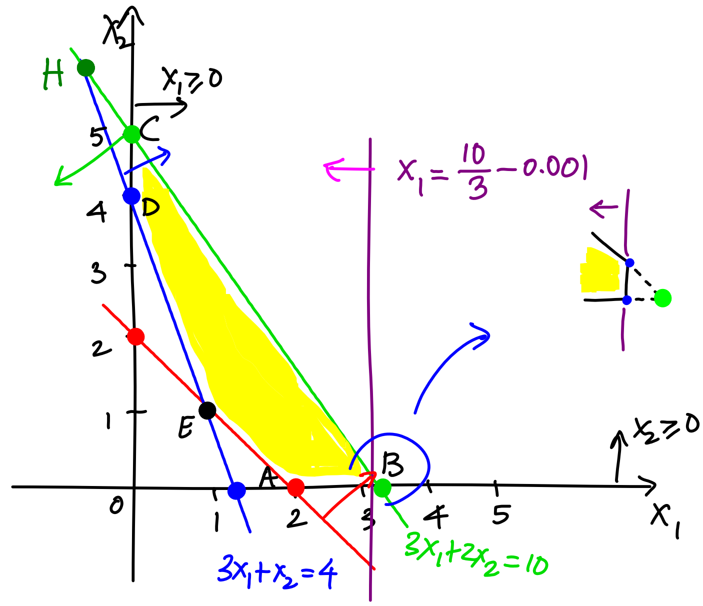
But if we replace $x_1 + x_2 + 2x_3 = 1$ now with two inequalities $x_1 + x_2 + 2x_3 \geq 1$ and $x_1 + x_2 + 2x_3 \leq 1$, B is degenerate again!
So, $B(0,0,1)$ is a degenerate bfs of

$$P = \{\bar{x} \in \mathbb{R}^3 \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 \geq 1,\ x_1 + x_2 + 2x_3 \leq 1,\ x_1 \geq 0,\ x_3 \geq 0\}.$$

constraints active at B

If permitted, we could avoid degeneracy by perturbing some of the constraint a tiny bit.

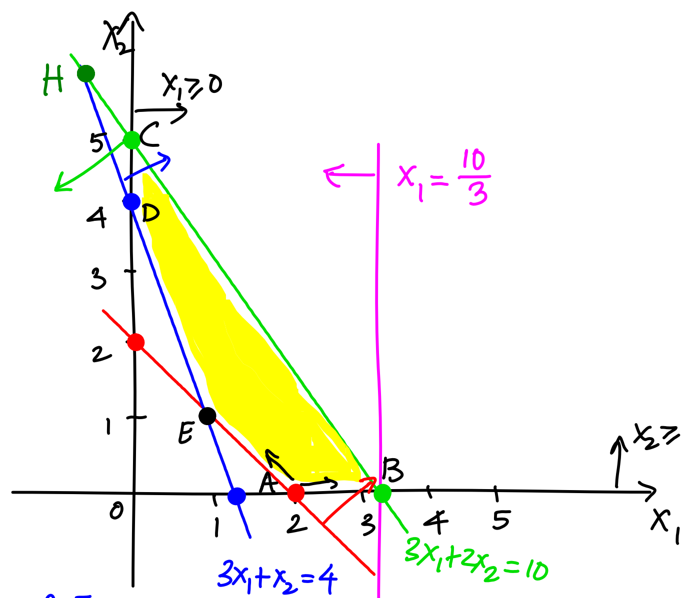
So, $x_1 \geq \frac{10}{3}$ could be replaced by $x_1 \geq \frac{10}{3} - 0.001$.



But whether we could do such minor modifications will depend very much on the specific application in question!

Let's go back to our example, and identify degenerate bfs's:

Recall that a basic solution $\bar{x} \in P$ in standard form is degenerate if more than $n-m$ x_j's are zero.



$$
\begin{aligned}
x_1 + x_2 - x_3 \quad &= 2 \\
3x_1 + x_2 \quad - x_4 &= 4 \\
3x_1 + 2x_2 \quad + x_5 &= 10 \\
x_1 \quad + x_6 &= 10/3
\end{aligned}
$$

$x_j \geq 0 \ \forall j$

original A

original $\bar{b}$

$$
A = \underbrace{\begin{bmatrix} \boxed{\begin{matrix} 1 & 1 & -1 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}}_{4 \times 6}
\qquad
\bar{b} = \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 4 \\ 10 \end{matrix}} \\ 10/3 \end{bmatrix}
$$

$\text{rank}(A) = m = 4.$

Indeed, rank$(A) = 4 = m$ now. So we are still in the original setting of standard form polyhedron. Let's figure out the bfs corresponding to the vertex B. Notice that $\{x_1, x_3, x_4\}$ are all basic, i.e., >0 at B. We need one more x_j out of $x_2, x_5, x_6)$ to complete a basis. Which one could we select?

With $B(1)=1, B(2)=2, B(3)=3, B(4)=4$, we get $B = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & 1 & 0 & -1 \\ 3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $\det(B) = -2$,

So B is indeed invertible. We get $\bar{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = B^{-1}\bar{b} = \begin{bmatrix} 10/3 \\ 0 \\ 4/3 \\ 6 \end{bmatrix}$.

In fact, we could pick any one of $\{x_2, x_5, x_6\}$ with $\{x_1, x_3, x_4\}$ to get a basis here. Check the Matlab session for details. In each case, we get $\underline{3\ x_j\text{'s}}$ set at 0. Recall that $n-m = 6-4 = 2$ here, thus showing
including the non-basic x_j's
that more than $n-m$ x_j's are zero at the degenerate bfs.

Qn. If there are $k > n$ active constraints, is it always true that there are $\binom{k}{k-n} = \binom{k}{n}$ different bases that lead to the same degenerate bfs? (will be in next homework)