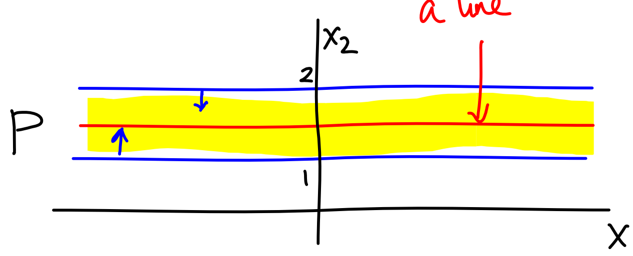
MATH464 — Lecture 11 (02/14/2023)

Today: * properties of polyhedra
* feasible and basic directions

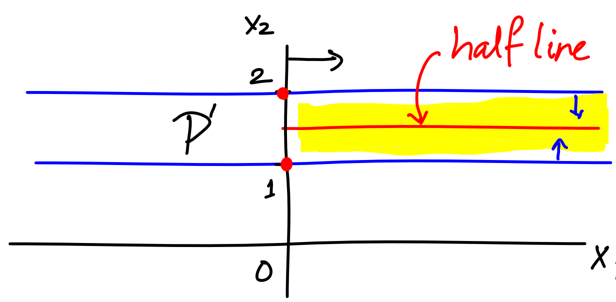
We have seen that vertices were optimal in many of the LP instances we solved. Indeed, we are building up the machinery to talk about the Simplex algorithm, which will move from one bfs to an adjacent bfs in each step. But we first study some properties of the polyhedron in general.

Does every polyhedron have an extreme point?

$P = \{\bar{x} \in \mathbb{R}^2 \mid 1 \le x_2 \le 2\}$ has no corner points.



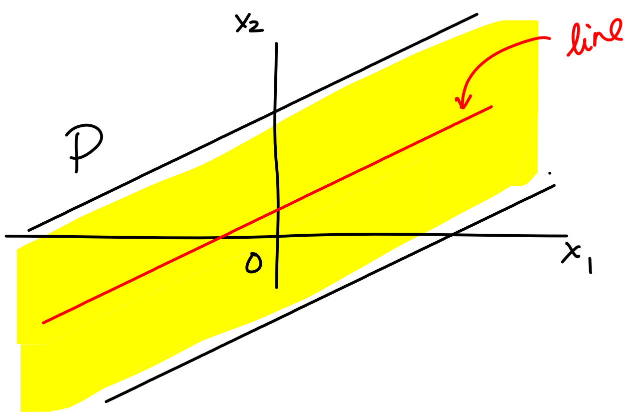
But if we were to add $x_1 \geqslant 0$ to make in P', we get two corner points.

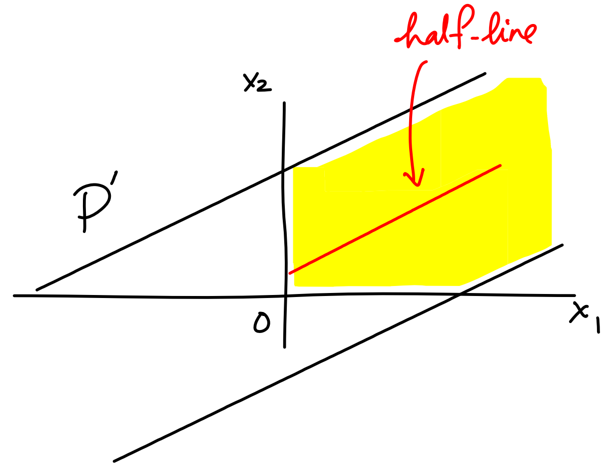


$P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \geqslant \bar{b}\}$ with $A_{m\times n}$ and $m < n$ cannot have any basic solutions, and hence cannot have any bfs's!

We make the following observation. P has a line, which is parallel to the two constraint lines (which themselves are parallel to each other). But P' (with $x_1 \geqslant 0$) can have only a half line (and not a line).

Intuitively, P is "corner-free" if it contains a line, which extends without limit in either direction. Another example: P on the left a line, and hence has no vertices.

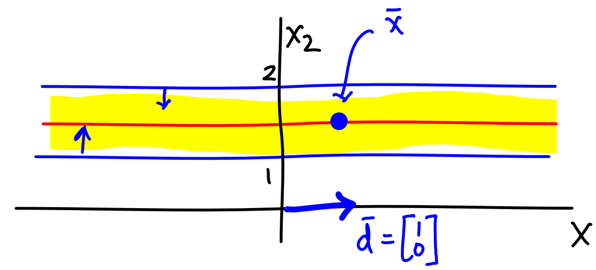




But P does not contain a line once $x_j \geqslant 0$ is added (to get P').

We formalize the notion of a set containing a line, and the intuition that a polyhedron without a line has a vertex.

Def $P \subset \mathbb{R}^n$ contains a line if there exists some $\bar{x} \in P$ and a direction $\bar{d} \in \mathbb{R}^n$, $\bar{d} \neq \bar{0}$, such that $\bar{x} + \lambda \bar{d} \in P \;\forall\; \lambda \in \mathbb{R}$.



Theorem 2.6 BT-ILO Let $P = \{\bar{x} \in \mathbb{R}^n \mid \bar{a}_i^T \bar{x} \geqslant b_i,\ i = 1, \dots, m\}$, $P \neq \emptyset$.

$m \geqslant n$

The following statements are equivalent.

(i) P has at least one extreme point.

(ii) P does not contain a line.

(iii) There are n vectors in $\{\bar{a}_1, \dots, \bar{a}_m\}$ which are LI.

We had already noted that if $m < n$, P has no bfs.

We immediately get the following corollary.

BT-ILO Corollary 2.2 Every nonempty bounded polyhedron, and every polyhedron in standard form, has at least one bfs.

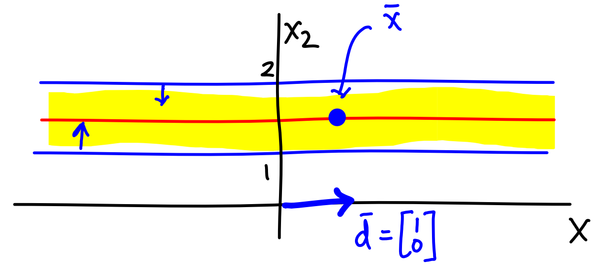
If P is bounded, it cannot contain a line, so (ii) holds.

Notice that $\bar{a}_i^T \bar{x} \geq b_i$, $i=1,\dots,m$ above include $x_j \geq 0 \; \forall j$. Indeed, as soon as we have added $x_j \geq 0 \; \forall j$, we do have n $\bar{a}_i$'s that are LI — the unit vectors corresponding to $x_j \geq 0$, i.e., $\bar{a}_i = \bar{e}_i$, the i^{th} unit vector. Hence (iii) holds.

Qn. If P has no corner points, does the LP

$$\min \{ \bar{c}^T \bar{x} \mid \bar{x} \in P \}$$

have any optimal solutions?



e.g., $\min \{ x_2 \mid 1 \leq x_2 \leq 2 \}$ has alternative optimal solutions (any point on the line $x_2 = 1$ is optimal).

But $\min \{ x_1 + x_2 \mid 1 \leq x_2 \leq 2 \}$ is an unbounded LP.

At the same time, we could make the following statement.

If P has no corner points, then the LP **cannot** have a unique optimal solution. The next theorem formalizes the reverse implication.

BT-ILO Theorem 2.7 Consider $\min\{\bar{c}^T\bar{x} \mid \bar{x} \in P\}$.
If P has at least one extreme point, and if the LP has an optimal solution, then there exists an extreme point which is an optimal solution.

We could get a slightly more general result, which specifies what happens when there is no optimal solution.

BT-ILO Theorem 2.8 Consider $\min\{\bar{c}^T\bar{x} \mid \bar{x} \in P\}$, where P is a polyhedron with at least one extreme point. Then the optimal cost is $-\infty$ or there exists an extreme point which is optimal.

See BT-ILO for the proofs.

Notice that the above theorems cover the case of polyhedra with at least one extreme point. But what about polyhedra without extreme points? Indeed, we can generalize the above theorem to get the following corollary. In particular, recall $\min\{x_2 \mid 1 \le x_2 \le 2\}$.

(Corollary 2.3 BT-ILO) Consider $\min\{\bar{c}^T\bar{x} \mid \bar{x} \in P\}$, where P is a nonempty polyhedron. Then the optimal cost is $-\infty$, or the exists an optimal solution.

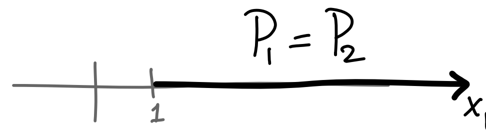
This result might not hold for non-linear problems.

consider $\left\{ \begin{array}{l} \min \frac{1}{x} \\ \text{s.t. } x \geq 1 \end{array} \right\}$ There is no optimal solution here, but the optimal cost is not $-\infty$.

We are now ready to present the simplex method to solve LPs in general dimensions. We will present this algorithm for LPs in standard form. In this context, we consider one more aspect of the standard form.

Qn. Does the "shape" of a polyhedron change when we convert it to standard form?

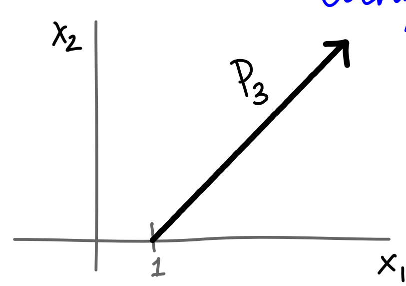
Consider $P_1 = \{x_1 \in \mathbb{R} \mid x_1 \geq 1\}$.



We add nonnegativity to get
$P_2 = \{x_1 \in \mathbb{R} \mid x_1 \geq 1, x_1 \geq 0\}$. Notice that $P_2 = P_1$.

Indeed, $x_1 \geq 0$ is redundant since $x_1 \geq 1$ is given. But we're just following the procedure here (to eventually convert P_1 to standard form).

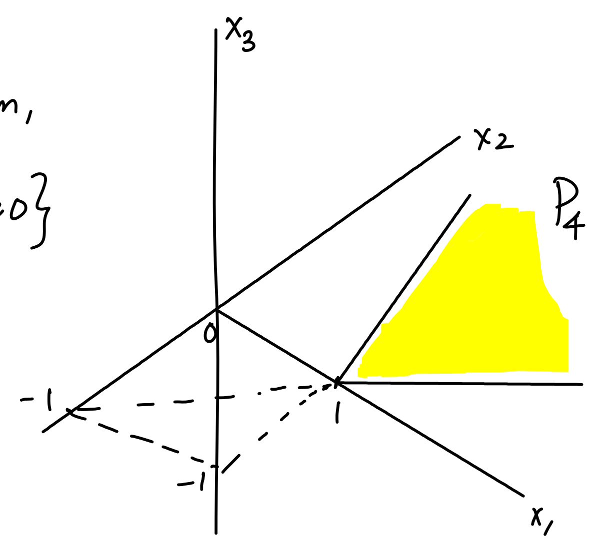
We convert P_2 to standard form to get
$P_3 = \{\bar{x} \in \mathbb{R}^2 \mid x_1 - x_2 = 1, x_1, x_2 \geq 0\}$.



P_3 looks very much like P_1 (or P_2). They are all half-lines (or rays).

Now, if we convert P_1 to standard form,
we get $P_4 = \{\bar{x} \in \mathbb{R}^3 \mid \underbrace{(x_1 - x_2)}_{= x_1 \text{ in } P_1} - x_3 = 1, x_j \geq 0\}$

x_1 urs



P_4 is a portion of a plane in the nonnegative orthant in $\mathbb{R}^3$. Notice the similarity to P_1, P_2, and P_3.

Simplex Method (Chapter 3 in BT-ILO)

The simplex method generalizes the graphical solution method in 2D to higher dimensions. It moves from one bfs (vertex) to an adjacent bfs (vertex) where the objective function improves.

We will describe the method for an LP in standard form where $A \in \mathbb{R}^{m \times n}$, $\bar{b} \in \mathbb{R}^m$, $\text{rank}(A) = m$, $m \leq n$.

$$\left\{ \begin{array}{ll} \min & \bar{c}^T \bar{x} \\ \text{s.t.} & A\bar{x} = \bar{b} \\ & \bar{x} \geq \bar{0} \end{array} \right\}.$$

We want to define **optimality conditions** for $\bar{x} \in P$. If these conditions are satisfied, then $\bar{x}$ is an optimal solution.

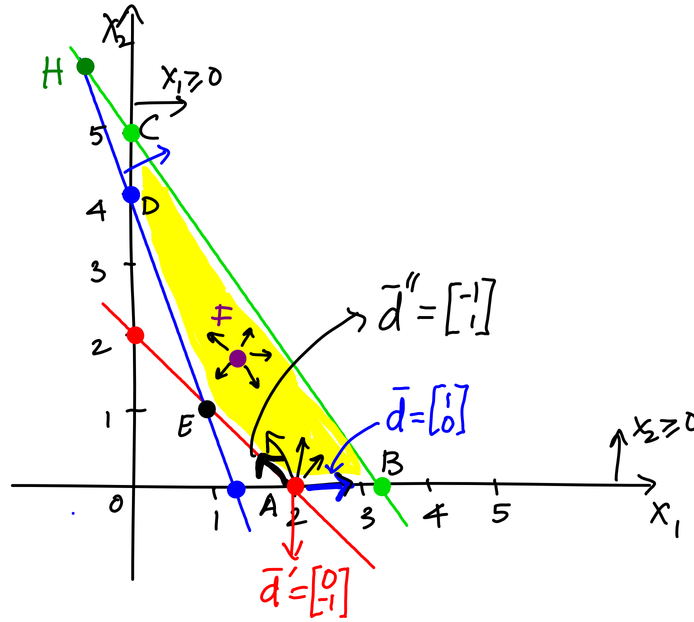
Algorithm: Given some $\bar{x} \in P$, check optimality conditions. If they are not satisfied, we search "nearby" to see if we can improve, i.e., decrease, $\bar{c}^T \bar{x}$.

In 1D calculus, the optimality conditions for $x \in \mathbb{R}$ to be a local minimum of $f(x)$ are $f'(x) = 0$ and $f''(x) > 0$. If $f(x)$ is a convex function, these conditions also guarantee that x is a global minimum.

For LPs, since $f(\bar{x}) = \bar{c}^T \bar{x}$ is linear, and since P is a polyhedron (hence convex), a local optimum is also a global optimum.

When searching "nearby", we want to make sure we always stay feasible, i.e., we do not want to go outside the feasible region.

Suppose we are at $A(2,0)$. We can consider directions to move. If we move straight down, i.e., along $\bar{d}' = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, we will go outside the feasible region. But $d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a good direction — we can move right all the way up to $B(10/3, 0)$.



In general, from $\bar{x}$, we consider $\bar{x} + \theta\bar{d}$ for $\theta > 0$ to move along $\bar{d}$. We want $\bar{x} + \theta\bar{d} \in P$.

Thus, no $\theta > 0$ exists for $\bar{d}'$, while any $0 < \theta \leq \frac{4}{3}$ works for moving along $\bar{d}$. Similarly, if we move Northwest, i.e., along $\bar{d}' = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we can go all the way to $E(1,1)$. So any $0 < \theta \leq 1$ works. Further, all directions "in between" $d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $d'' = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are good — see figure.

If F is in the interior of P, then any $\bar{d}$ is a good direction (see figure).

We formalize this notion of a "good" direction. (in the next lecture...)