MATH 464 – Lecture 13 (02/21/2023)

Today: * LP optimality conditions
* Details of the Simplex method

Recall: Reduced costs: $\bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1} A$ (for LP in standard form: $\min\{\bar{c}^T\bar{x} \mid A\bar{x}=\bar{b}, \bar{x}\geqslant\bar{0}\}$ with $A\in\mathbb{R}^{m\times n}$, $\text{rank}(A)=m$, $m\leq n$.

BT-ILO Theorem 3.1 $\bar{x}$ is a bfs, B: basis matrix, $\bar{c}'$ reduced costs.

(a) If $\bar{c}'\geqslant\bar{0}$, then $\bar{x}$ is optimal

(b) If $\bar{x}$ is nondegenerate and optimal, then $\bar{c}'\geqslant\bar{0}$.

Proof (a) Let $\bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1} A \geqslant \bar{0}^T$. We want to show $\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y}$ $\forall \bar{y}\in P$ (i.e., $\bar{x}$ is optimal).

Let $\bar{y}\in P$ be an arbitrary feasible point. So, $A\bar{y}=\bar{b}, \bar{y}\geqslant\bar{0}$.

Let $\bar{d}=\bar{y}-\bar{x}$. Also, $A\bar{x}=\bar{b}$, $\bar{x}\geqslant\bar{0}$ (as $\bar{x}$ is a bfs).

Want to prove $\bar{c}^T\bar{d}\geqslant\bar{0}$

$$\Rightarrow A\bar{d} = A(\bar{y}-\bar{x}) = \bar{0}.$$

$$\Rightarrow B\bar{d}_B + N\bar{d}_N = \bar{0} \quad \Rightarrow B^{-1}\Big(B\bar{d}_B + \sum_{i\in\mathcal{N}} A_i d_i = \bar{0}\Big).$$

$\mathcal{N}$ is the set of non-basic indices ($x_j=0$ $\forall j\in\mathcal{N}$).

$\mathcal{B}$ is the set of basis indices, i.e., $\mathcal{B}=\{B(1),\ldots,B(m)\}$.

So, $\mathcal{B}\cup\mathcal{N} = \{1,\ldots,n\}$.

$$\Rightarrow \bar{d}_B + \sum_{i\in\mathcal{N}} B^{-1}A_i d_i = \bar{0} \quad \Rightarrow \bar{d}_B = -\sum_{i\in\mathcal{N}} B^{-1}A_i d_i.$$

So, $\bar{c}^T \bar{d} = \bar{c}_B^T \bar{d}_B + \bar{c}_N^T \bar{d}_N = \bar{c}_B^T \bar{d}_B + \sum_{i \in N} c_i d_i$

$$= \sum_{i \in N} c_i d_i - \sum_{i \in N} \bar{c}_B^T B^{-1} A_i d_i \qquad \bar{d}_B = -\sum_{i \in N} B^{-1} A_i d_i$$

$$= \sum_{i \in N} (c_i - \bar{c}_B^T B^{-1} A_i) d_i$$

$$= \sum_{i \in N} \underbrace{c_i'}_{\geq 0} d_i, \quad \text{as } c_i' = c_i - \bar{c}_B^T B^{-1} A_i, \text{ the } i^{th} \text{ reduced cost.}$$

We will be done if we can show $d_i \geq 0 \;\forall\, i \in N$, as $c_i' \geq 0$ is already given, and then we get $\bar{c}^T \bar{d} \geq 0 \Rightarrow \bar{c}^T \bar{y} \geq \bar{c}^T \bar{x}$.

We have $d_i = y_i - x_i \;\forall\, i \in N$. But $x_i = 0 \;\forall\, i \in N$ (as $\bar{x}$ is a bfs).

Also, $y_i \geq 0$, as $\bar{y} \in P$ and hence $\bar{y} \geq \bar{0}$.

$\Rightarrow d_i \geq 0 \;\forall\, i \in N. \;\Rightarrow\; \bar{c}^T \bar{d} \geq 0$, i.e., $\bar{c}^T \bar{x} \leq \bar{c}^T \bar{y}$.

$\Rightarrow \bar{x}$ is optimal.

Check BT-ILO for proof of statement (b). $\square$

Equivalent definition of optimality conditions

→ we combine feasibility and optimality

A basis matrix B is optimal if

(a) $B^{-1}\bar{b} \geq \bar{0}$, and (feasibity)

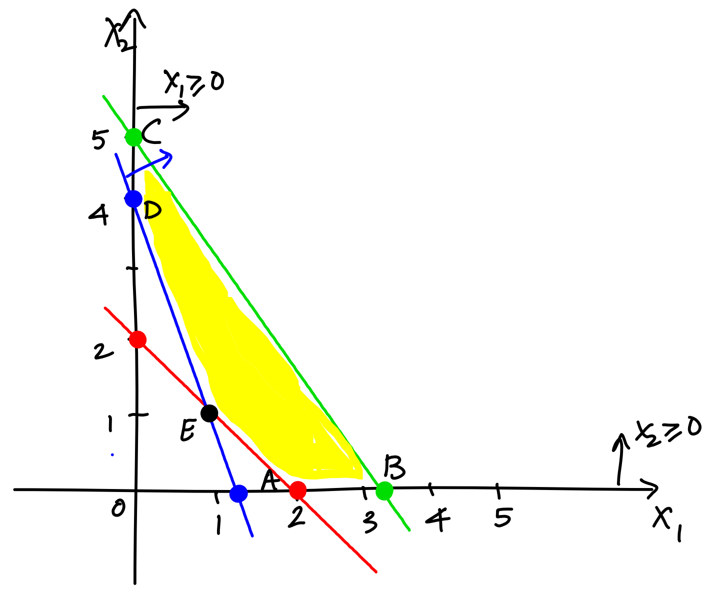
(b) $\bar{c}'^{T} = \bar{c}^{T} - \bar{c}_B^{T} B^{-1} A \geq \bar{0}$ (optimality).

At $E(1,1)$, basis is

$\mathcal{B} = \{B(1), B(2), B(3)\} = \{1, 2, 5\}$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \qquad B^{-1}\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \geq \bar{0}$$

$$\bar{c}' = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} \geq \bar{0}.$$ So, E is optimal.



We already saw (in Lecture 12) that at $A(2,0)$, the basis is the basis is $\mathcal{B} = \{B(1), B(2), B(3)\} = \{1, 4, 5\}$, giving

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad B^{-1}\bar{b} = \begin{array}{c} 1 \\ 4 \\ 5 \end{array}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \geq \bar{0},$$ but

$$\bar{c}' = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \not\geq \bar{0}.$$ Hence A is feasible but not optimal.

Recall: optimality conditions for basis matrix B:

(a) $B^{-1}\bar{b} \geq \bar{0}$ (feasibility)

(b) $\bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1} A \geq \bar{0}^T$ (optimality).

Simplex Method

How to move to a "better" bfs, given a "good" direction.

Assume all bfs's are nondegenerate. We will deal with degeneracy later.

Let $\bar{x}$ be a bfs. If $\bar{c}' \geq \bar{0}$, $\bar{x}$ is optimal, and we can stop.

If $c_j' < 0$ for some $j \in \mathcal{N}$, then we can move along $\bar{d}$, the jth basic direction, and improve the objective function value. Recall that $\bar{d}$ is given by $\bar{d}_B = -B^{-1}A_j$, $d_j = 1$, $d_i = 0$, $i \in \mathcal{N}, i \neq j$.

If we move θ "units" along $\bar{d}$, x_j increases from 0 to $\theta > 0$. (as $\bar{x} \to \bar{x} + \theta\bar{d}$).

We say that x_j **enters the basis**.

$\mathcal{B} = \{B(1), \ldots, B(m)\}$, and $j \notin \mathcal{B}$ initially.

Subsequently some x_i for $i \in \mathcal{B}$, i.e., a basic variable, has to **leave the basis**, i.e., it becomes 0.

If $B(\ell) = i$, then $x_{B(\ell)} = 0$ in the next bfs.

Recall that the basis should have exactly m indices (out of $1, \ldots, n$). Since j just entered $\mathcal{B}$, one of the current indices should leave.

Which variable leaves?

$\bar{x} \longrightarrow \bar{x}+\theta\bar{d}$. As θ increases, $\bar{c}^T\bar{x}$ decreases. So, go as far as we can go, while staying feasible. So, set $\theta=\theta^*$, where

$$\theta^* = \max\{\theta \geq 0 \mid \bar{x}+\theta\bar{d} \in P\}$$

already satisfies $A(\bar{x}+\theta\bar{d})=\bar{b}$

We need to insure $\bar{x}+\theta\bar{d} \geq \bar{0}$ as well.

$x_i \longrightarrow x_i+\theta d_i \geq 0$:

For $i \in \mathcal{N}$, $x_j = \theta > 0$, $x_i = 0 \;\forall\, i \in \mathcal{N}, i \neq j$.

We need to make sure $x_i \geq 0 \;\forall\, i \in \mathcal{B}$. If $d_i \geq 0$, then $x_i+\theta d_i$ stays >0 (as $x_i>0$ to start with, and $\theta>0$).

So we look at the cases where $d_i<0$, for $i \in \mathcal{B}$. But overall,

$$x_i+\theta d_i \geq 0 \;\Rightarrow\; \theta \geq \frac{-x_i}{d_i},\; i \in \mathcal{B}.$$

If $d_i>0$ for some i, we get $\theta \geq$ (negative #), which is redundant.

(1) If $\bar{d} \geq \bar{0}$, then $\theta \to \infty$, and $\bar{c}^T\bar{x} \to -\infty$, i.e., the LP is unbounded (Case III).

(2) If $d_i<0$ for at least one i, then set

$$\theta^* = \min_{\{i \mid d_i<0\}} \left(\frac{-x_i}{d_i}\right)$$

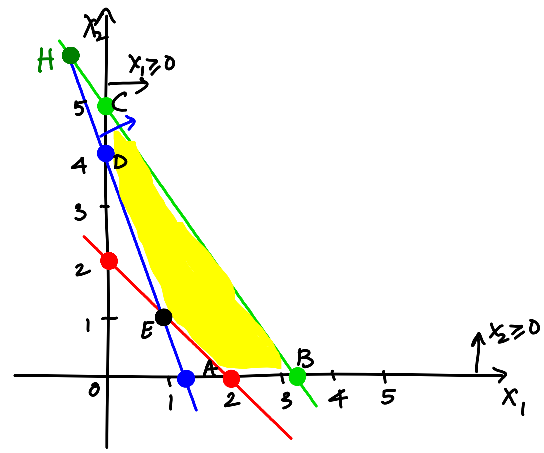
$$\theta^* = \min_{i \in \mathcal{B} \mid \underline{d_{B(i)}<0}} \left(\frac{-x_{B(i)}}{d_{B(i)}}\right)$$

this is termed the minimum-ratio or **min-ratio test**

it is important to consider the ratio $\frac{-x_i}{d_i}$ only when $d_i<0$ here.

Back to 2D Example

$\mathcal{B} = \{1,3,4\}, \; \mathcal{N} = \{2,5\}.$



At $B(10/3, 0)$, basis has $B(1)=1, \; B(2)=3, \; B(3)=4$.

$$\bar{c}' = \begin{bmatrix} 0 \\ -1/3 \\ 0 \\ 0 \\ -2/3 \end{bmatrix}, \text{ showing that the bfs } \bar{x} = \begin{bmatrix} 10/3 \\ 0 \\ 4/3 \\ 6 \\ 0 \end{bmatrix}$$

is not optimal. For $j=2$, the second basic direction is given by

$$\bar{d} = \begin{bmatrix} -2/3 \\ 1 \\ 1/3 \\ -1 \\ 0 \end{bmatrix}. \quad d_i < 0$$

Hence $\theta^* = \min\left\{ -\frac{10/3}{-2/3}, \; -\frac{6}{-1} \right\}$

$= \min\{5, 6\} = 5.$

We move $\theta^* = 5$ units along $\bar{d}$ to get to the bfs corresponding to $C(0,5)$.

$$\bar{x} + \theta^* \bar{d} = \begin{bmatrix} 10/3 \\ 0 \\ 4/3 \\ 6 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} -2/3 \\ 1 \\ 1/3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

point $C(0,5)$.