MATH 464 – Lecture 14 (02/23/2023)

Today: * an iteration of the simplex method
* proofs related to simplex method

An iteration of the Simplex Algorithms

We will describe how to find a starting bfs later on.

1. Start with $\mathcal{B} = \{B(1), \dots, B(m)\}$ and bfs $\bar{x}$.

2. Find $c_j' = c_j - \bar{c}_B^T B^{-1} A_j$ for $j \in \mathcal{N}$.
   if $c_j' \geq 0 \ \forall j \in \mathcal{N}$ then STOP; $\bar{x}$ is optimal
   else choose a j with $c_j' < 0$. → more on this choice later on!

3. Find $\bar{d}_B = -B^{-1} A_j$
   if $\bar{d}_B \geq \bar{0}$, then $\theta^* = \infty$, optimal cost $= -\infty$; STOP.
   LP is unbounded.

4. Some entry in $\bar{d}_B$ is < 0. Let
$$\theta^* = \min_{i \in \mathcal{B}, d_{B(i)} < 0} \left(\frac{-x_{B(i)}}{d_{B(i)}} \right) \text{ and let } \theta^* = \frac{-x_{B(\ell)}}{d_{B(\ell)}}$$
   where ℓ: index of the winner of the min-ratio test.

5. Replace $B(\ell)$ in $\mathcal{B}$ with j; → x_j enters the basis
   New basis $\mathcal{B}'$ is given by $B'(i) = \begin{cases} B(i), & i \neq \ell \\ j, & i = \ell. \end{cases}$

   New bfs $\bar{x}' = \bar{x} + \theta^* \bar{d}$ has $x_j' = \theta^*$, and
$$x'_{B(i)} = x_{B(i)} + \theta^* d_{B(i)}, \quad i \in \mathcal{B}, \ i \neq \ell;$$
$$x'_{B(\ell)} = 0.$$

We will prove $\bar{x}'$ is indeed a bfs!

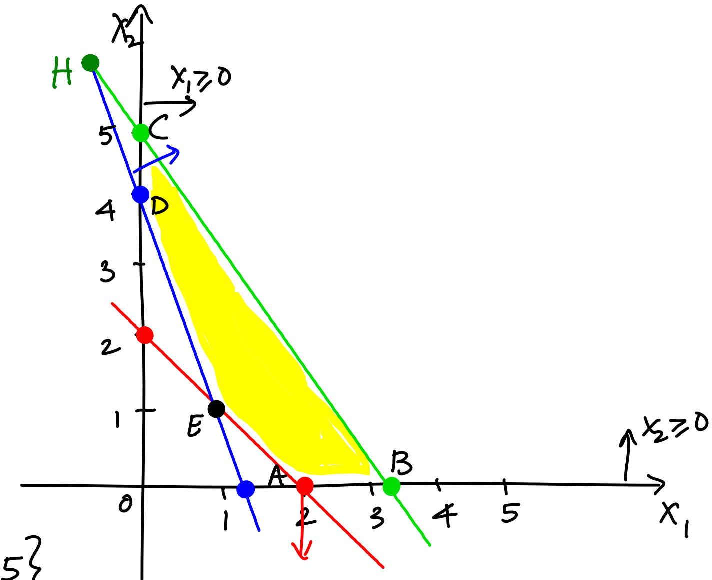
Illustration

Iteration 1

Let's start at $B(10/3, 0)$.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

$$\bar{c}^T = [2 \ 1 \ 0 \ 0 \ 0]$$



$$\bar{x} = \begin{bmatrix} 10/3 \\ 0 \\ 4/3 \\ 6 \\ 0 \end{bmatrix} \qquad \mathcal{B} = \{1, 3, 4\}, \ \mathcal{N} = \{2, 5\}$$

$$\bar{c}'_{\mathcal{N}} = \begin{array}{c} 2 \\ 5 \end{array} \begin{bmatrix} -1/3 \\ -2/3 \end{bmatrix}$$

Let's pick $j=2$: $\bar{d}_B = \begin{array}{c} 1 \\ 3 \\ 4 \end{array} \begin{bmatrix} -2/3 \\ 1/3 \\ -1 \end{bmatrix}$ $\qquad \theta^* = \min\left\{ \underbrace{-\frac{10/3}{-2/3}}_{\ell=1}, -\frac{6}{-1} \right\} = 5.$

$$\bar{x}' = \bar{x} + \theta^* \bar{d} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 1 \\ 0 \end{bmatrix} \longrightarrow C(0,5).$$

x_2 entered the basis, and x_1 left the basis. $\mathcal{B}' = \{2, 3, 4\}$.

Iteration 2

$\mathcal{B} = \{2, 3, 4\}$, $\mathcal{N} = \{1, 5\}$, $\bar{x} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{bmatrix} 0 \\ 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

Reduced costs $\bar{c}' = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \\ -1/2 \end{bmatrix}$. $\quad j = 5$ enters the basis now.

Notice that $x_5 = 0$ at $C(0,5)$; indeed $3x_1 + 2x_2 \leq 10$ is active at C.

$$\bar{d}_B = -B^{-1}A_2 = \begin{array}{c} 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} -\tfrac{1}{2} \\ -\tfrac{1}{2} \\ -\tfrac{1}{2} \end{bmatrix} \qquad \theta^* = \min\left\{ \frac{-5}{-\tfrac{1}{2}}, \frac{-3}{-\tfrac{1}{2}}, \frac{-1}{-\tfrac{1}{2}} \right\} = 2$$

$\ell = 3$ here.

So, $B(\ell) = 4$ leaves the basis, as $j = 5$ enters.

$$\bar{x}' = \bar{x} + \theta^* \bar{d} = \begin{bmatrix} 0 \\ 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

corresponds to $D(0,4)$

X_4 leaves the basis, X_5 enters; $\mathcal{B}' = \{2, 3, 5\}$.

Iteration 3

$$\mathcal{B} = \{2, 3, 5\},\ \mathcal{N} = \{1, 4\},\ \bar{x} = \begin{array}{c} \\ 2 \\ 3 \\ \\ 5 \end{array}\begin{bmatrix} 0 \\ 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}.$$

We get

$$\bar{c}' = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

$j = 1$ enters the basis.

$$\bar{d}_B = -B^{-1}A_1 = \begin{array}{c} 2 \\ 3 \\ 5 \end{array}\begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix} \qquad \theta^* = \min\left\{ \frac{-4}{-3}, \frac{-2}{-2} \right\} = 1,\ \ell = 2$$

$B(\ell) = 3$ leaves the basis. $\mathcal{B}' = \{2, 1, 5\}$, $\bar{x}' = \bar{x} + \theta^* \bar{d} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

corresponds to $E(1,1)$

Iteration 4

$$\mathcal{B} = \{2, 1, 5\},\ \mathcal{N} = \{3, 4\},\ \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

$$\bar{c}' = \begin{bmatrix} 0 \\ 0 \\ \tfrac{1}{2} \\ \tfrac{1}{2} \\ 0 \end{bmatrix} \geqslant \bar{0}.$$

Hence $\bar{x}$ is an optimal bfs.

Optimal cost $z^* = \bar{c}^T \bar{x} = 3$.

Correctness of the Simplex Method

We present some theorems that certify the correctness of the simplex method. We first show that the algorithm will terminate, and then present results certifying the structure of each solution visited along the way, i.e., that they're adjacent bfs's.

Q. Will the simplex method always terminate?

(BT-ILO Theorem 3.3) Let $P \neq \emptyset$, every bfs be nondegenerate. Then the simplex method terminates after a finite number of iterations with one of two outcomes:

(a) We get an optimal basis $\mathcal{B}$ and an optimal bfs $\bar{x}$; or

(b) We get a direction $\bar{d}$ such that $A\bar{d}=\bar{0}$, $\bar{d} \geq \bar{0}$, and $\bar{c}^T\bar{d} < 0$, showing that the optimal cost is $-\infty$, i.e., the LP is unbounded.

Proof

STOP in Steps 2 and 3 give the two outcomes.

In each iteration, $\theta^* \xrightarrow{\text{nondegeneracy assumption}}$ $\theta^* > 0$, $\bar{c}^T\bar{d} < 0$, so the cost strictly decreases in each step. Hence we do not visit a bfs more than once. But there are only finitely many bfs's — at most $\binom{n}{m}$. Hence the algorithm must terminate after a finite number of steps.

Q. In Step 5, we set $\bar{x}' = \bar{x} + \theta^* \bar{d}$. How can we be sure that $\bar{x}'$ is indeed a bfs?

BT-ILO Theorem 3.2

(a) The columns $A_{B(i)}$, $i \neq l$, and A_j are LI, and hence B', the new basis matrix, is indeed a basis matrix.

(b) $\bar{x}' = \bar{x} + \theta^* \bar{d}$ is a bfs associated with B'.

Proof

We have $B' = \begin{bmatrix} | & | & & | & & | & & | \\ A_{B(1)} & A_{B(2)} & \cdots & A_{B(l-1)} & A_j & A_{B(l+1)} & \cdots & A_{B(m)} \\ | & | & & | & & | & & | \end{bmatrix}$

and $B = \begin{bmatrix} | & | & & | & & | & & | \\ A_{B(1)} & A_{B(2)} & \cdots & A_{B(l-1)} & A_{B(l)} & A_{B(l+1)} & \cdots & A_{B(m)} \\ | & | & & | & & | & & | \end{bmatrix}$.

We want to show columns of B' are LI. Instead, we look at the columns of $B^{-1}B'$. Notice that $B^{-1}B = I$. Hence

$$B^{-1}B' = [\,\bar{e}_1 \ \bar{e}_2 \ \cdots \ \bar{e}_{l-1} \ B^{-1}A_j \ \bar{e}_{l+1} \cdots e_m\,], \quad \text{where}$$

$\bar{e}_i$ is the i^{th} unit m-vector. l^{th} entry.

But $B^{-1}A_j = -\bar{d}_B$. We note that $d_{B(l)} \neq 0$, as l is the index of the winner of the min-ratio test. Hence $-\bar{d}_B$ is LI from the remaining $\bar{e}_i$ vectors, which are LI by themselves. Hence the columns of $B^{-1}B'$, and hence that of B', are LI.

(b) We have $A\bar{x}' = \bar{b}$ and $\bar{x}' \geqslant \bar{0}$ (feasibility), and $\bar{x}'$ is associated with the matrix B' which is a basis matrix.

Hence $\bar{x}'$ is a bfs. $\square$