MATH 464 – Lecture 15 (02/28/2023)

Today: * details of simplex method
* revised simplex method

Details of the Simplex Method

Pivot Selection

Choices in steps 2 and 5: choose a j s.t. $c_j' < 0$ and $\ell = \underset{\{i \in B \mid d_{B(i)} < 0\}}{\operatorname{argmin}} \left(\frac{-x_{B(i)}}{d_{B(i)}} \right)$

Here are a few options.

(a) Pick j such that $c_j' < 0$ and c_j' is the most negative reduced cost among $j \in \mathcal{N}$ (fastest rate of decrease of $\bar{c}^T\bar{x}$).

(b) Pick j such that $c_j' < 0$ and $|c_j'|\theta^*$ is largest. (Largest net decrease in $\bar{c}^T\bar{x}$).

Option (a) picks the direction of steepest descent, but we might not be able to move too far in the steepest direction.

Option (b) has more foresight than option (a), but also requires more computation. In (b), we have to find the basic direction $(\bar{d}_B = -B^{-1}A_j)$ for each j, and then do the min-ratio test for each j (to find the θ^*).

(c) For really large problems, calculate c_j's one at a time until a $c_j' < 0$ is found, then just go with that j.

We cannot afford to compute the entire $\bar{c}'$ vector – so, just compute c_j' one at a time.

A naive implementation

(one iteration)

1. Given a bfs $\bar{x}$, basis $\mathcal{B} = \{B(1), \dots, B(m)\}$, basis matrix B, solve $\bar{p}^T B = \bar{c}_B^T$ to get $\bar{p}^T = \bar{c}_B^T B^{-1}$, the **simplex multipliers**.

   → same as $B^T \bar{p} = \bar{c}_B$; but we use $\bar{p}^T$ directly afterward.

2. Find reduced costs for non-basic variables

$$\bar{c}_{\mathcal{N}}'^T = \bar{c}_{\mathcal{N}}^T - \underbrace{\bar{c}_B^T B^{-1}}_{\bar{p}^T} A_{\mathcal{N}} = \bar{c}_{\mathcal{N}}^T - \bar{p}^T A_{\mathcal{N}}.$$

$\mathcal{N}$ – nonbasic indices

If $\bar{c}_{\mathcal{N}}'^T \geq \bar{0}^T$ current bfs is optimal; STOP.

Else pick a j such that $c_j' < 0$.

systems of linear equations!

3. Solve $B\bar{d}_B = -A_j$ to get $\bar{d}_B = -B^{-1}A_j$ ← jth basic direction

If $\bar{d}_B \geq \bar{0}$ STOP. $\bar{c}^T\bar{x} \to -\infty$

else find

$$\theta^* = \min_{\{i \in \mathcal{B} \mid d_{B(i)} < 0\}} \left(\frac{-x_{B(i)}}{d_{B(i)}}\right) = \left(\frac{-x_{B(\ell)}}{d_{B(\ell)}}\right).$$

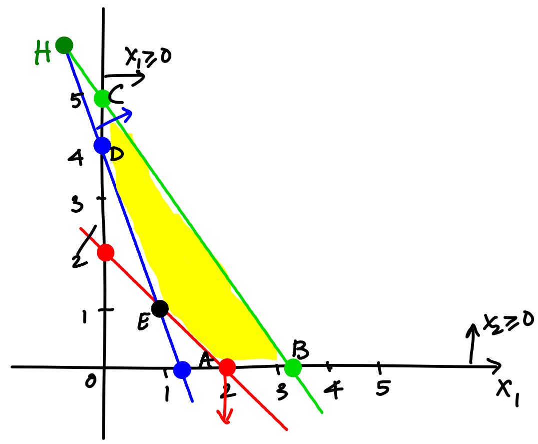
4. Change basis: replace $B(\ell)$ with j, update bfs to $\bar{x}'$ such that $x_j' = \theta^*$, $x_{B(i)}' = x_{B(i)} + \theta^* d_{B(i)}$ $\forall i \neq j, i \in \mathcal{B}$.

Solving the two systems of linear equations repeatedly could also prove costly! Hence the "naive" implementation.

Revised Simplex Method

From one iteration to the next, B changes by only one column. If we know B^{-1} from previous iteration, we could use it to find new B^{-1} (or $(B')^{-1}$) quickly. So we store B^{-1}, and update it efficiently in each step.

$$\begin{aligned}
\min\quad & 2x_1 + x_2 \\
\text{s.t.}\quad & x_1 + x_2 - x_3 = 2 \\
& 3x_1 + x_2 - x_4 = 4 \\
& 3x_1 + 2x_2 + x_5 = 10 \\
& x_j \geq 0 \ \forall j
\end{aligned}$$



Correspondences between bfs's and corner points

pt	Basis $\mathcal{B}$: $B(1)$	$B(2)$	$B(3)$	B	B^{-1}	x_1	x_2	x_3	x_4	x_5
A	1	4	5	$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$	2	0	0	2	4
B	1	3	4	$\begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & -1 \\ 3 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1/3 \\ -1 & 0 & 1/3 \\ 0 & -1 & 1 \end{bmatrix}$	$10/3$	0	$4/3$	6	0
C	2	3	4	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1/2 \\ -1 & 0 & 1/2 \\ 0 & -1 & 1/2 \end{bmatrix}$	0	5	3	1	0
D	2	3	5	$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$	0	4	2	0	2
E	1	2	5	$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & 1/2 & 0 \\ 3/2 & -1/2 & 0 \\ -3/2 & -1/2 & 1 \end{bmatrix}$	1	1	0	0	5

From one iteration to next, the basis matrix B changes by only one column. So, if we know B^{-1} from the previous iteration, we can use it to find the new B^{-1} efficiently.

Q. given B^{-1}, how do we find new B^{-1} (or $(B')^{-1}$)?

We know $B^{-1}B = I$, i.e., $\bar{e}_i$: i^{th} unit m-vector

$$B^{-1}[A_{B(1)}\ A_{B(2)} \cdots A_{B(\ell)} \cdots A_{B(m)}] = [\bar{e}_1\ \bar{e}_2 \cdots \bar{e}_\ell \cdots \bar{e}_m]$$

New basis matrix $B' = [A_{B(2)} \cdots A_j \cdots A_{B(m)}]$ x_j entering the basis; replacing $x_{B(\ell)}$; $\{B(\ell) \to j\}$ in $\mathcal{B}$

$$\Rightarrow B^{-1}B' = [\bar{e}_1\ \bar{e}_2 \cdots B^{-1}A_j \cdots \bar{e}_m]; \text{ recall } \bar{d}_B = -B^{-1}A_j.$$

$$\Rightarrow B^{-1}B' = [\bar{e}_1\ \bar{e}_2 \cdots -\bar{d}_B \cdots \bar{e}_m].$$

Recall EROs to invert an $n \times n$ matrix $[A|I] \xrightarrow{\text{EROs}} [I|A^{-1}]$

Following this idea, we need a few more EROs to convert $-\bar{d}_B$ to $\bar{e}_\ell$.

Let $QB^{-1}B' = I$, where Q represents the EROs on $B^{-1}B'$ that convert it to I.

$B^{-1}B'$ is almost I, except of $-\bar{d}_B$ in the ℓ^{th} column. Q represents the EROs that convert $-\bar{d}_B$ to $\bar{e}_\ell$.

$$\Rightarrow QB^{-1} = (B')^{-1}$$

We can start with $[B^{-1}|-\bar{d}_B]$, and do EROs to convert $-\bar{d}_B$ to $\bar{e}_\ell$. We will get $(B')^{-1}$ sitting in place of B^{-1}.

Back to the example

When we go from $B(10/3, 0)$ to $C(0,5)$, the basis matrix sees only one column changed:

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & -1 \\ 3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\mathcal{B} = \{\overset{\ell}{1}, 3, 4\}$$

$$\mathcal{B}' = \{\underset{j}{2}, 3, 4\}$$

We have $j=2$ (entering) and $\ell=1$ (leaving).

At $B(10/3, 0)$, $B^{-1} = \begin{bmatrix} 0 & 0 & 1/3 \\ -1 & 0 & 1/3 \\ 0 & -1 & 1 \end{bmatrix}$. $\Rightarrow \bar{d}_B = -B^{-1}A_j$

$$\Rightarrow -\bar{d}_B = B^{-1}A_j = \begin{bmatrix} 0 & 0 & 1/3 \\ -1 & 0 & 1/3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ 1 \end{bmatrix}.$$

$$[B^{-1}|-\bar{d}_B] \xrightarrow{\text{EROs}} [(B')^{-1}|\bar{e}_\ell] \qquad \ell = 1 \text{ here.}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1/3 & 2/3 \\ -1 & 0 & 1/3 & -1/3 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[\substack{\text{and then} \\ R_2 + \frac{1}{3}R_1 \\ R_3 - R_1}]{R_1 \times 3/2} \left[\begin{array}{ccc|c} 0 & 0 & 1/2 & 1 \\ -1 & 0 & 1/2 & 0 \\ 0 & -1 & 1/2 & 0 \end{array}\right] \Rightarrow (B')^{-1} = \underbrace{\begin{bmatrix} 0 & 0 & 1/2 \\ -1 & 0 & 1/2 \\ 0 & -1 & 1/2 \end{bmatrix}}$$

this is indeed the $(B')^{-1}$ at the bfs corresponding to $C(0,5)$.

If we were to invert B' from scratch, we would be solving m such systems, and not just one!

One iteration of the revised simplex method

1. Start with basis $\mathcal{B} = \{B(1), \dots, B(m)\}$, basis matrix B, bfs $\bar{x}$.
   $\bar{x} = B^{-1}\bar{b}$; we assume B^{-1} is known.

2. Find $\bar{c}_{\mathcal{N}}^{\prime T} = \bar{c}_{\mathcal{N}}^{T} - \bar{c}_B^T B^{-1} A_{\mathcal{N}}$.

3. If $\bar{c}_{\mathcal{N}}^{\prime T} \geq \bar{0}$, then current bfs is optimal. STOP.
   Else pick non-basic x_j to enter the basis (with $c_j' < 0$).

4. Find $\bar{d}_B = -B^{-1}A_j$.
   If $\bar{d}_B \geq \bar{0}$, STOP. LP is unbounded, $\bar{c}^T\bar{x} \to -\infty$.

5. Choose $x_{B(\ell)}$ to leave the current basis such that

$$\left(\frac{-x_{B(\ell)}}{d_{B(\ell)}}\right) = \theta^* = \min_{\{d_{B(i)}<0,\ i\in\mathcal{B}\}} \left(\frac{-x_{B(i)}}{d_{B(i)}}\right).$$

   New bfs is $\bar{x}' = \bar{x} + \theta^*\bar{d}$.

6. Find new B^{-1} (or $(B')^{-1}$) using EROs:

$$\left[B^{-1} \,\middle|\, -\bar{d}_B\right] \xrightarrow{\text{EROs}} \left[(B')^{-1} \,\middle|\, \bar{e}_\ell\right].$$