MATH464 – Lecture 16 (03/02/2023)

Today: * Example for revised simplex method
* full tableau implementation

Problems from Hw6

5. **Exercise 2.16** Consider the set $S = \{\mathbf{x} \in \Re^n \mid x_1 = \cdots = x_{n-1} = 0,\ 0 \leq x_n \leq 1\}$. Could this be the feasible set of a problem in standard form?

P in standard form is $\{\bar{x} \in \mathbb{R}^n \mid A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0}\}$, with rank$(A) = m$, $m \leq n$.

The given set (let's call it S) is obviously not in standard form. The question is whether this set could describe a set that is in standard form.

You should assume $n \geq 2$.

Note that $\bar{x} = \bar{0} \in S$. So, if $S = P$, then $\bar{0} \in P$ as well.

$\Rightarrow A(\bar{0}) = \bar{b} \Rightarrow \bar{b} = \bar{0}$ in the standard form.

with $\bar{b} = \bar{0}$, if $\bar{x} \in P$, $\lambda\bar{x} \in P \ \forall \lambda \geq 0$

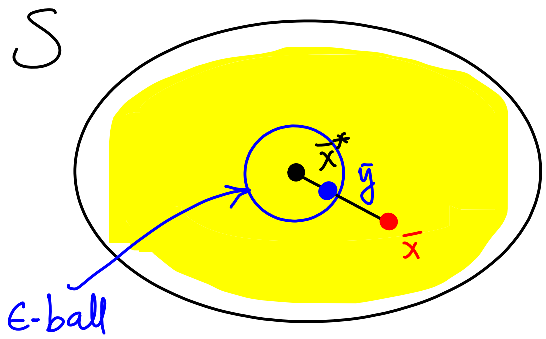
$A(\lambda\bar{x}) = \lambda A(\bar{x}) = \lambda(\bar{b}) = \lambda\bar{0} = \bar{0}$. & $\lambda\bar{x} \geq \bar{0}$.

So $\lambda > 1$ is also good.

But $\lambda\bar{x}_n \not\leq 1$ for $\lambda > 1$.

So $\lambda\bar{x} \notin S$ for $\lambda > 1$, giving an obstruction for $S = P$.

6. Problem 3.6. Show that a local minimum $\bar{x}^*$ of a convex function over a convex set S is the global minimum.



$\bar{x}^*$ is a local optimum of $f \Rightarrow$
$\exists\, \epsilon > 0$, s.t.
$f(\bar{x}^*) \leq f(\bar{x}) \;\; \forall\, \bar{x}$ s.t. $\|\bar{x} - \bar{x}^*\| \leq \epsilon$

Need to prove $\bar{x}^*$ is global minimum, i.e., $f(\bar{x}^*) \leq f(\bar{x}) \;\; \forall\, \bar{x} \in S$
(and not just inside the ϵ-ball around $\bar{x}^*$).

Proof by contradiction: Assume $\exists\, \bar{x}$ outside the ϵ-ball centered at $\bar{x}^*$ with $f(\bar{x}) < f(\bar{x}^*)$, i.e., $\bar{x}^*$ is **not** a global minimum.

Consider a point $\bar{y}$ on the line segment connecting $\bar{x}^*$ and $\bar{x}$ that is within the ϵ-ball @ $\bar{x}^*$, i.e., $\|\bar{y} - \bar{x}^*\| < \epsilon$.

$$\Rightarrow \; f(\bar{x}^*) \leq f(\bar{y}) \qquad \text{——— (1)}$$

Q. Why are guaranteed to find such a $\bar{y}$?

$\bar{y}$ is a point on the line segment joining $\bar{x}^*$ and $\bar{x}$

$$\Rightarrow \; \bar{y} = \lambda \bar{x}^* + (1-\lambda)\bar{x} \quad \text{for some } \lambda \in (0,1).$$

Q. Why do we choose λ strictly between 0 and 1?

$$\Rightarrow \; f(\bar{y}) = f(\lambda \bar{x}^* + (1-\lambda)\bar{x}) \leq \lambda f(\bar{x}^*) + (1-\lambda)\bar{x} \qquad \text{——— (2)}$$

why?

Combine (1) & (2) to get a contradiction...

Hw7: AMPL problems. You **must** separate the model from data. A good rule to follow: do **not** include any actual numbers, i.e., data in your model file.

An example for the Revised Simplex Method

Recall how we maintain and update B^{-1}:

$$[B^{-1} \mid -d_B] \xrightarrow{\text{EROs}} [(B')^{-1} \mid \bar{e}_\ell]$$

Solve using the revised simplex method.

$$\begin{aligned} \min\ & -x_1 - x_2 \\ \text{s.t.}\ & 2x_1 + x_2 \le 4 \\ & 3x_1 + 5x_2 \le 15 \\ & x_1, x_2 \ge 0 \end{aligned} \quad\quad \begin{aligned} \min\ & -x_1 - x_2 \\ \text{s.t.}\ & 2x_1 + x_2 + x_3 = 4 \\ & 3x_1 + 5x_2 + x_4 = 15 \\ & x_j \ge 0\ \forall j \end{aligned}$$

$$\bar{c}^T = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} -1 & -1 & 0 & 0 \end{bmatrix} \end{array} \quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}.$$

$m = 2,\ n = 4$

Iteration 1

$B(1) = 3,\ B(2) = 4 \quad \mathcal{B} = \{3, 4\}, \mathcal{N} = \{1, 2\}. \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B^{-1} = B = I.$

$\bar{c}_B^T = [0\ \ 0]. \quad \bar{c}_{\mathcal{N}}'^T = \bar{c}_{\mathcal{N}}^T - \underbrace{\cancel{\bar{c}_B^T} B^{-1} N}_{=0} = \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ [-1 \ \ -1] \end{array} \quad j = 1 \ \ (x_1 \text{ enters}).$

$$\bar{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = B^{-1}\bar{b} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}. \quad \bar{d}_B = -B^{-1}A_1 = -\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$\theta^* = \min\left\{ \frac{-4}{-2}\checkmark, \frac{-15}{-3} \right\} = 2 \quad \ell = 1 \quad B(1) = 3 \text{ leaves the basis } (x_3 \text{ leaves}).$$

$$\mathcal{B}' = \{1, 4\}. \quad \bar{x}' = \bar{x} + \theta^* \bar{d} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 15 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 9 \end{bmatrix}.$$

$$[B^{-1} \mid -\bar{d}_B] \xrightarrow{\text{EROs}} [(B')^{-1} \mid \bar{e}_\ell] \quad \bar{e}_\ell = \bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ here.}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow[\text{then } R_3 - 3R_1]{R_1 \times \frac{1}{2}} \left[\begin{array}{cc|c} \frac{1}{2} & 0 & 1 \\ -\frac{3}{2} & 1 & 0 \end{array}\right] \implies \text{new } B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}.$$

Iteration 2

$\mathcal{B} = \{1,4\}$, $\mathcal{N} = \{2,3\}$. $\bar{c}_B^T = [-1 \;\; 0]$

$$\bar{c}_{\mathcal{N}}'^T = \bar{c}_{\mathcal{N}}^T - \bar{c}_B^T B^{-1} N = \begin{array}{cc} 2 & 3 \\ \end{array} [-1 \;\; 0] - \underbrace{[-1 \;\; 0]\begin{bmatrix} 1/2 & 0 \\ -3/2 & 1 \end{bmatrix}}_{[-1/2 \;\; 0]} \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$$

$$= \begin{array}{cc} 2 & 3 \\ \end{array} [-1/2 \;\; 1/2] \quad j = 2 \; (x_2 \text{ enters}). \qquad \underbrace{[-1/2 \;\; 0]\begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}}_{[-1/2 \; -1/2]}$$

no tie to break here!

$$\bar{d}_B = -B^{-1}A_2 = -\begin{bmatrix} 1/2 & 0 \\ -3/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{array}{c} 1 \\ 4 \end{array}\begin{bmatrix} -1/2 \\ -7/2 \end{bmatrix} \qquad \bar{x} = \begin{array}{c} 1 \\ \\ \\ 4 \end{array}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

$$\theta^* = \min\left\{ \frac{-2}{-1/2}, \frac{-9}{-7/2} \right\} = \frac{18}{7}, \quad l = 2 \text{ here. } B(2) = 4 \text{ leaves.}$$

$\mathcal{B}' = \{1,2\}$ (new basis). $\bar{x}' = \bar{x} + \theta^* \bar{d} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 9 \end{bmatrix} + \frac{18}{7}\begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -7/2 \end{bmatrix} = \begin{bmatrix} 5/7 \\ 18/7 \\ 0 \\ 0 \end{bmatrix}.$

Update B^{-1}:

$$\left[\begin{array}{cc|c} 1/2 & 0 & 1/2 \\ -3/2 & 1 & 7/2 \end{array}\right] \xrightarrow[\text{then } R_1 - \frac{1}{2}R_2]{R_2 \times 2/7} \left[\begin{array}{cc|c} 5/7 & -1/7 & 0 \\ -3/7 & 2/7 & 1 \end{array}\right] \qquad \text{New } B^{-1} = \begin{bmatrix} 5/7 & -1/7 \\ -3/7 & 2/7 \end{bmatrix}.$$

Iteration 3

$\mathcal{B} = \{1,2\}$, $\mathcal{N} = \{3,4\}$. $\bar{c}_B^T = \begin{array}{cc} 1 & 2 \\ \end{array}[-1 \;\; -1]$ $\quad \bar{c}_{\mathcal{N}}^T = \begin{array}{cc} 3 & 4 \\ \end{array}[0 \;\; 0]$

$$\bar{c}_{\mathcal{N}}'^T = \bar{c}_{\mathcal{N}}^T - \bar{c}_B^T B^{-1} N = [0 \;\; 0] - \underbrace{[-1 \; -1]\begin{bmatrix} 5/7 & -1/7 \\ -3/7 & 2/7 \end{bmatrix}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= [2/7 \;\; 1/7] \geq 0$$

$\bar{x} = \begin{bmatrix} 5/7 \\ 18/7 \\ 0 \\ 0 \end{bmatrix}$ is an optimal bfs, with the optimal objective function value $z^* = \bar{c}^T \bar{x} = \frac{-23}{7}$.

Full-Tableau Implementation (LP in standard form)

For large LPs, the revised simplex method is the efficient way to solve them. But we will also introduce a compact way of representing the numbers, and implementing all the computations – the tableau simplex method.

We maintain and update $[\bar{x}_B \mid B^{-1}A]$ or $B^{-1}[\bar{b} \mid A]$. This is an $m \times (n+1)$ matrix, or table, called the **simplex tableau**.

$$\begin{array}{ccccccc} 0 & & 1 & 2 & \cdots & n & \\ \end{array}$$
$$\left[B^{-1}\bar{b} \;\middle|\; B^{-1}A_1 \;\; B^{-1}A_2 \cdots B^{-1}A_n \right]$$

Nothing new here!
$B^{-1}(A\bar{x} = \bar{b}) \Longrightarrow$
$(B^{-1}A)\,\bar{x} = B^{-1}\bar{b}$
$[B^{-1}\bar{b} \mid B^{-1}A]$

0^{th} column: values of the basic variables
i^{th} column: has $B^{-1}A_i$, for $i = 1, \dots, n$.

If x_j is entering, the column having $B^{-1}A_j = -\bar{d}_B$ is the **pivot column**.

If $x_{B(\ell)}$ leaves the basis $(1 \le \ell \le n)$, then the ℓ^{th} row is the **pivot row**. The (ℓ, j)-element is the **pivot element** or just the pivot.

We include the costs $\bar{c}^T$ and the objective function $z = \bar{c}^T\bar{x}$ at the top as the zero-th row.

$\bar{c}'$ (reduced cost vector)

$$\begin{array}{c|c|c|}
 & 0 & 1 \; 2 \; \cdots \; n \\
\hline
0 & -\bar{c}_B^T B^{-1}\bar{b} & \bar{c}^T - \bar{c}_B^T B^{-1} A \\
\hline
1 & & \\
2 & B^{-1}\bar{b} & B^{-1}A \\
\vdots & & \\
m & & \\
\hline
\end{array}
=
\begin{array}{c|c|ccccc|}
 & 0 & 1 & 2 & \cdots\; j & \cdots & n \\
\hline
0 & -\bar{c}_B^T B^{-1}\bar{b} & c_1' & c_2' & \cdots\; c_j' & \cdots & c_n' \\
\hline
1 & x_{B(1)} & & & & & \\
2 & x_{B(2)} & B^{-1}A_1 & B^{-1}A_2 & \cdots\; B^{-1}A_j & \cdots & B^{-1}A_n \\
\vdots & \vdots & & & & & \\
m & x_{B(m)} & & & & & \\
\hline
\end{array}$$

$-z = -\bar{c}_B^T \bar{x}_B$