MATH 464 – Lecture 17 (03/07/2023)

Today: * Hw7 problems
* tableau implementation of simplex

Hw 7

Exercise 3.2 (Optimality conditions) Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ over a polyhedron P. Prove the following:

(a) A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}'\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

A B

(b) A feasible solution $\mathbf{x}$ is the unique optimal solution if and only if $\mathbf{c}'\mathbf{d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$.

A iff B $\equiv$ A $\Rightarrow$ B and B $\Rightarrow$ A

($\Rightarrow$) $\bar{x}$ is optimal. $\Rightarrow$ $\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y}$ $\forall\, \bar{y} \in P$.

Want to show: $\bar{c}^T\bar{d} \geq 0$ $\forall$ feasible direction $\bar{d}$ at $\bar{x}$.

$\bar{d}$ is a feasible direction at $\bar{x}$ $\Rightarrow$ $\exists\, \theta > 0$ s.t. $\bar{x} + \theta\bar{d} \in P$

$\bar{c}^T(\bar{y} - \bar{x}) \geq 0$

Argue that every feasible direction at $\bar{x}$ can be written as $\bar{y} - \bar{x}$.

This could help in the reverse direction below as well!

($\Leftarrow$) $\bar{c}^T\bar{d} \geq 0$ $\forall$ feasible direction $\bar{d}$ at $\bar{x}$

$\exists\, \theta > 0$ s.t. $\bar{x} + \theta\bar{d} \in P$

from this, derive $\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y}$

try to use $\bar{x} + \theta\bar{d}$ as $\bar{y}$

Exercise 3.3 Let $\mathbf{x}$ be an element of the standard form polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Prove that a vector $\mathbf{d} \in \Re^n$ is a feasible direction at $\mathbf{x}$ if and only if $\mathbf{A}\mathbf{d} = \mathbf{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$.

A

B

$A \Leftrightarrow B$

$(\Rightarrow)$ $\bar{d}$ is a feasible direction @ $\bar{x}$: $\exists\, \theta > 0$ s.t. $\bar{x} + \theta\bar{d} \in P$

$\Rightarrow A(\bar{x} + \theta\bar{d}) = \bar{b}$, $\bar{x} + \theta\bar{d} \geq \bar{0}$

But $A\bar{x} = \bar{b} \Rightarrow \theta A\bar{d} = \bar{0} \Rightarrow A\bar{d} = \bar{0}$.

$\bar{x} + \theta\bar{d} \geq \bar{0}$, $x_i = 0 \Rightarrow \theta d_i \geq 0$; $\theta > 0 \Rightarrow d_i \geq 0$.

$(\Leftarrow)$ Given $A\bar{d} = \bar{0}$, $d_i \geq 0$ when $x_i = 0$, show (for $\bar{x} \in P$)

$\exists\, \theta > 0$ s.t. $A(\bar{x} + \theta\bar{d}) = \bar{b}$ and $\bar{x} + \theta\bar{d} \geq \bar{0}$.

We already have $A\bar{x} = \bar{b}$, and now have $A\bar{d} = \bar{0}$.

$\Rightarrow A(\bar{x} + \theta\bar{d}) = \bar{b}$

To show $\exists\, \theta > 0$ s.t. $\bar{x} + \theta\bar{d} \geq \bar{0}$,

think about the min-ratio test!

Exercise 3.6 (Conditions for a unique optimum) Let $\mathbf{x}$ be a basic feasible solution associated with some basis matrix $\mathbf{B}$. Prove the following:

(a) If the reduced cost of every nonbasic variable is positive, then $\mathbf{x}$ is the unique optimal solution.

Use arguments similar to those used for Problem 3.1 (in Homework 6).

Full Tableau Implementation of the Simplex Method

reduced costs $\bar{c}'^T$

$$
\begin{array}{c|c|c}
 & 0 & 1\ 2\ \cdots\ n \\
\hline
0 & -\bar{c}_B^T B^{-1} \bar{b} & \bar{c}^T - \bar{c}_B^T B^{-1} A \\
\hline
1,2,\ldots,m & B^{-1}\bar{b} & B^{-1}A
\end{array}
=
\begin{array}{c|c|cccccc}
 & 0 & 1 & 2 & \cdots & j & \cdots & n \\
\hline
0 & -\bar{c}_B^T B^{-1} \bar{b} & c_1' & c_2' & \cdots & c_j' & \cdots & c_n' \\
\hline
1 & x_{B(1)} & & & & & & \\
2 & x_{B(2)} & B^{-1}A_1 & B^{-1}A_2 & \cdots & B^{-1}A_j & \cdots & B^{-1}A_n \\
\vdots & \vdots & & & & & & \\
m & x_{B(m)} & & & & & &
\end{array}
$$

$$-z = -\bar{c}_B^T x_B$$

$$
\left.
\begin{aligned}
\min \quad & -x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \le 4 \\
& 3x_1 + 5x_2 \le 15 \\
& x_1, x_2 \ge 0
\end{aligned}
\right\}
\quad
\left.
\begin{aligned}
\min \quad & -x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 + x_3 = 4 \\
& 3x_1 + 5x_2 + x_4 = 15 \\
& x_j \ge 0 \ \forall j
\end{aligned}
\right\}
\quad
\begin{array}{l}
\bar{c}^T = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 \end{bmatrix} \end{array} \\[2ex]
A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}. \\[2ex]
m = 2,\ n = 4
\end{array}
$$

	z	x_1	x_2	x_3	x_4	min-ratio candidates	
	0	-1	-1	0	0		$R_0 + R_1'$
$x_3 =$	4	2	1	1	0	2	$R_1(\frac{1}{2}) = R_1'$
$x_4 =$	15	3	5	0	1	5	$R_2 - 3R_1'$
	2	0	$-\frac{1}{2}$	$\frac{1}{2}$	0		
$x_1 =$	2	1	$\frac{1}{2}$	$\frac{1}{2}$	0	4	
$x_4 =$	9	0	$\frac{7}{2}$	$-\frac{3}{2}$	1	$\frac{18}{7}$	
	$\frac{23}{7}$	0	0	$\frac{2}{7}$	$\frac{1}{7}$		
$x_1 =$	$\frac{5}{7}$	1	0	$\frac{5}{7}$	$-\frac{1}{7}$		
$x_2 =$	$\frac{18}{7}$	0	1	$-\frac{3}{7}$	$\frac{2}{7}$		

B^{-1}

We first scale the pivot row so that the pivot is 1. We use replacement EROs to zero out the rest of the pivot column.

Optimal solution is $x_1 = 5/7$, $x_2 = 18/7$, with $z^* = -\frac{23}{7}$.

: We indicate the pivot by circling the entry.

Since we maintain $B^{-1}A$, if some columns of A form I, the identity matrix, B^{-1} will be sitting in those columns.

We add the labels of basic variables on the left of each tableau, mainly to improve readability. One could identify the basic variables by spotting the unit vectors among the variable columns.

Why use $-\bar{c}_B^T \bar{x}_B$ in Row-0, Column-0 of the tableau?

If all constraints were $\leq$, we could choose the m slack variables in our starting bfs, and all original $x_j = 0$ to start with.
Hence $z = \bar{c}^T \bar{x} = 0$ to start with.

Row-0: $[0 \mid \bar{c}^T] - \bar{g}^T [\bar{b} \mid A]$ where $\bar{g}^T = \bar{c}_B^T B^{-1}$ are multipliers.

i.e., a combination of m replacement EROs.

In columns 1 to n, we get $\bar{c}^T - \bar{g}^T A = \bar{c}^T - \bar{c}_B^T B^{-1} A = \bar{c}'$.

In column 0, we get $z \leftarrow z - \bar{g}^T \bar{b} = z - \bar{c}_B^T B^{-1} \bar{b}$

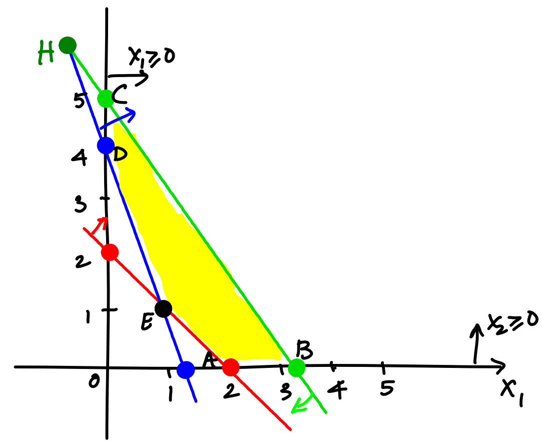
"replace"

$= z - \bar{c}_B^T \bar{x}_B$

So, if we start with $z=0$, we get $-\bar{c}_B^T \bar{x}_B$ in Row-0, Column-0.

Back to our favorite example

$$\begin{aligned} \min \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 = 2 \\ & 3x_1 + x_2 - x_4 = 4 \\ & 3x_1 + 2x_2 + x_5 = 10 \\ & x_j \geq 0 \ \forall j \end{aligned}$$



We start at the bfs $\equiv B(10/3, 0)$, i.e., $\mathcal{B} = \{1, 3, 4\}$.

We will talk about how to find a starting bfs in general later on.

We move from B to A to E.

See the course web page for the Matlab session. (for the first iteration).

We'll finish the calculations in the next lecture...