MATH 464 – Lecture 19 (03/21/2023)

Today: * lexicographic pivoting rule
* big-M method

Recall: Lexicographic order: $\bar{u} \overset{L}{>} \bar{v}$ if the first nonzero entry of $\bar{u}-\bar{v}$ is positive.

Lexicographic Pivoting Rule (for tableau simplex method)

1. Choice of entering variable is arbitrary (as long as $c_j' < 0$).
2. For each a_{ij} in the pivot column j $(B^{-1}A_j)$ that is > 0, we divide the entire Row-i including column-0, by a_{ij} so that we get a 1 in place of a_{ij}. We then choose the lexicographically smallest row, say Row-ℓ. The corresponding basic variable $x_{B(\ell)}$ leaves the basis.

Back to the cycling example:

Objective function does not change for iterations 1-3, just as before. But it improves in iterations 4 and 5, giving the optimal solution as $z^* = -5/4$ and $x_1 = 1$, $x_3 = 1$, $x_5 = 3/4$.

See the course web page for the Matlab session:
http://www.math.wsu.edu/faculty/bkrishna/FilesMath464/S23/Software/Lec19_03212023_Session.txt

Bland's rule (min-index rule)

1. Choose smallest j with $c_j' < 0$ as the pivot column (x_j enters).
2. If tied for leaving variable, choose the variable with the smallest index i to leave.

We will come back to Bland's rule later.

Finding Initial bfs

1. Two-phase simplex
2. big-M method (combines the two phases of Two-phase simplex).

$$\left.\begin{array}{ll} \min & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}=\bar{b} \\ & \bar{x}\geq\bar{0} \end{array}\right\} \qquad \begin{array}{ll} \min & \bar{c}^T\bar{x} + M\sum_{i=1}^{m} y_i \\ \text{s.t.} & A\bar{x} + I\bar{y} = \bar{b} \\ & \bar{x}\geq\bar{0},\ \bar{y}\geq\bar{0} \end{array} \qquad \bar{y}\in\mathbb{R}^m$$

We assume $\bar{b}\geq\bar{0}$. If some $b_i<0$ to start with, scale that constraint i by -1, so that new $b_i>0$ (before adding $I\bar{y}$).

The y_i's are called **artificial variables**, one per constraint. They are artificial as their purpose is only to provide a starting bfs.

$M \rightarrow$ "big-M" acts like $+\infty$, can be considered a huge positive number. We can compare expressions involving M (unlike ones using ∞):

$$3M+2-(4M-10) = -M+12$$

$$-M+10{,}000 < 2M-8.$$

Since the min-objective function has My_i term for each i, it helps to set $y_i=0\ \forall i$, if possible. Thus, if we obtain an optimal solution with $y_i=0\ \forall i$, the corresponding x values are an optimal solution to the original LP. But if any $y_i>0$ in the optimal solution, the objective function value is essentially $+\infty$, indicating the original LP is infeasible.

Let's try the big-M method on our favorite LP:

$$\begin{aligned} \min\ & 2x_1+x_2 \\ \text{s.t. } & x_1+x_2-x_3 = 2 \quad x_6 \\ & 3x_1+x_2-x_4 = 4 \quad x_7 \\ & 3x_1+2x_2+x_5 = 10 \quad x_8 \\ & x_j \geq 0 \ \forall j \end{aligned}$$

→ instead of y_1, y_2, y_3, we use x_6, x_7, x_8.

$$\begin{aligned} \min\ & 2x_1+x_2 + Mx_6+Mx_7+Mx_8 \\ \text{s.t. } & x_1+x_2-x_3+x_6 = 2 \\ & 3x_1+x_2-x_4+x_7 = 4 \\ & 3x_1+2x_2+x_5+x_8 = 10 \\ & x_j \geq 0 \ \forall j \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, \quad \bar{c}^T = [2\ 1\ 0\ 0\ 0\ M\ M\ M].$$

We can choose $\{x_6, x_7, x_8\}$ as the starting bfs. → the artificial vars, in general. $\bar{c}_B^T = [M\ M\ M]$.

$$B = I = B^{-1}. \quad \bar{x}_B = \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix} = B^{-1}\bar{b} = \bar{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.$$

$$\begin{aligned} \bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1} A &= [2\ \ 1\ \ 0\ \ 0\ \ 0\ \ M\ \ M\ \ M] - [7M\ \ 4M\ \ {-M}\ \ {-M}\ \ M\ \ M\ \ M\ \ M] \\ &= [-7M+2\ \ \ {-4M}+1\ \ \ M\ \ \ M\ \ \ {-M}\ \ \ 0\ \ \ 0\ \ \ 0] \end{aligned}$$

$$-\bar{c}_B^T \bar{x}_B = -16M.$$

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