MATH 464 – Lecture 20 (03/22/2018)

Today: * big-M method to detect infeasibility
* one full example
* revised simplex v/s tableau simplex

Detecting infeasibility

$$\begin{array}{ll} \min & 2x_1+x_2 \\ \text{s.t.} & x_1+x_2=4 \quad x_3 \\ & 2x_1+2x_2=10 \quad x_4 \\ & x_1,x_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \min & 2x_1+x_2+Mx_3+Mx_4 \\ \text{s.t.} & x_1+x_2+x_3 \qquad = 4 \\ & 2x_1+2x_2 \qquad +x_4 = 10 \\ & x_j \geq 0 \ \forall j \end{array}$$

$A=\begin{bmatrix}1&1&1&0\\2&2&0&1\end{bmatrix}$ $\bar{b}=\begin{bmatrix}4\\10\end{bmatrix}$, $\bar{c}^T=[2\ 1\ M\ M]$. Starting basis $\mathcal{B}=\{3,4\}$.

$\Rightarrow \bar{c}_B^T=[M\ M]$.

$B=B^{-1}=I$, $\bar{x}_B=\bar{b}$

$$\bar{c}'^T=\bar{c}^T-\bar{c}_B^TB^{-1}A$$
$$=[2\ 1\ M\ M]-[M\ M]\begin{bmatrix}1&1&1&0\\2&2&0&1\end{bmatrix}$$
$$=[-3M+2\ \ -3M+1\ \ 0\ \ 0].$$

$z=\bar{c}_B^T\bar{x}_B=[M\ M]\begin{bmatrix}4\\10\end{bmatrix}=14M.$

	$-14M$	$-3M+2$	$-3M+1$	0	0
$x_3=$	4	1	1	1	0
$x_4=$	10	2	2	0	1
	$-2M-4$	1	0	$3M-1$	0
$x_2=$	4	1	1	1	0
$x_4=$	2	0	0	-2	1

$R_0-(-3M+1)R_1$

→ optimal!

The optimal solution is $x_2=4$, $x_4=2$, with $z^*=2M+4$. Since an artificial variable (x_4) is >0 (basic), the original LP is infeasible.

Caution!: An artificial variable that is basic, but still $=0$ does not indicate infeasibility of original LP!

Two-phase Method

Given $\left\{\begin{array}{l}\min\ \bar{c}^T\bar{x} \\ \text{s.t. } A\bar{x}=\bar{b} \ \ (\bar{b}\geq\bar{0}) \\ \qquad \bar{x}\geq\bar{0}\end{array}\right\}$ we write the following LP:

$$\left.\begin{array}{ll}\min & \bar{1}^T\bar{y}=\sum_{i=1}^{m} y_i \\ \text{s.t.} & A\bar{x}+I\bar{y}=\bar{b} \\ & \bar{x},\bar{y}\geq\bar{0}\end{array}\right\} \text{Phase 1 LP.}$$

↳ is guaranteed to be feasible!

$\bar{1}$: vector of 1's.

Here, y_i is the artificial variable for constraint i. Assuming $\bar{b}\geq\bar{0}$, which can be ensured before standardizing the LP by scaling any constraint with $b_i<0$ by -1, $\{\bar{y}=\bar{b},\ \bar{x}=\bar{0}\}$ is a feasible solution. Thus, the Phase 1-LP is feasible. Further, it is guaranteed to have an optimal solution: the minimum value of $\sum_{i=1}^{m} y_i$ is 0, when $y_i=0\ \forall i$, as $y_i\geq 0\ \forall i$.

If the optimal objective function value of the phase-1 LP is zero, i.e., $y_i=0\ \forall i$ in the optimal solution, then the original LP is feasible. The corresponding $\bar{x}$ variables give a starting bfs for the Phase 2 LP, which goes back to $\min\{\bar{c}^T\bar{x} \mid A\bar{x}=\bar{b},\ \bar{x}\geq 0\}$.

On the other hand, if any $y_i>0$ in the optimal solution of the Phase 1 LP, the original LP is infeasible.

Phase-1 LP might be a better option than the big-M method if we're interested in only checking whether the original LP is feasible.

A full example

$$\max \bar{c}^T\bar{x} \equiv \min -\bar{c}^T\bar{x}$$

$$\begin{aligned}
\max\ & z = -2x_1 + 3x_2 \\
\text{s.t. } & x_1 + 3x_2 \le 9 \\
& -(2x_1 + 5x_2 \ge -6) \\
& x_2 \ge 1 \\
& x_1 \text{ urs},\ x_2 \ge 0
\end{aligned}$$

$x_1 \to (x_1 - x_3)$

so that rhs is $\ge 0 \Rightarrow -2x_1 - 5x_2 \le 6$

Steps:

1. Change objective to min, if needed.
2. Scale constraints with $b_i < 0$ by -1.
3. Take care of urs and ≤ 0 vars.
4. Convert to standard form.
5. Add artificial vars as needed.
6. Proceed with simplex method.

$$\begin{aligned}
\min\ z &= 2(x_1 - x_3) - 3x_2 + Mx_7 \\
x_1 - x_3 + 3x_2 + x_4 &= 9 \\
-2x_1 + 2x_3 - 5x_2 + x_5 &= 6 \\
x_2 - x_6 + x_7 &= 1
\end{aligned}$$

We could add artificial variables for constraints 1 and 2 as well; but if a slack var is present, we might as well use them in the starting basis!

For large LP instances with tens of thousands of constraints, most of which are $\le$, it is wasteful to add artifical variables unless necessary — we could use the slack variables in the starting bfs!

$$\bar{c}^T = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \left[\begin{array}{ccccccc} 2 & -3 & -2 & 0 & 0 & 0 & M \end{array}\right] \end{array}$$

$B = I = B^{-1}$ in starting tableau with $\mathcal{B} = \{4, 5, 7\}$, $B^{-1} = I$.

$$\bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1} A \qquad \bar{c}_B = [0\ 0\ M], \quad \bar{x}_B = B\bar{b} = \bar{b} = \begin{bmatrix} 9 \\ 6 \\ 1 \end{bmatrix}.$$

$$-\bar{z} = -\bar{c}_B^T \bar{x}_B = -M.$$

We'll calculate this vector directly in Matlab!

See Matlab session at

http://www.math.wsu.edu/faculty/bkrishna/FilesMath464/S23/Software/Lec20_03232023_Session.txt

Optimal solution of the transformed LP: $x_2 = 24$, $x_3 = 63$, $x_6 = 23$, $z^* = -198$.

For the original LP, we had $x_1 = x_1 - x_3$, and had changed max $\bar{c}^T\bar{x}$ to min $-\bar{c}^T\bar{x}$. Hence the optimal solution to the original LP is $x_1 = -63$, $x_2 = 24$ $(x_6 = 23)$, with $z^* = 198$.

Comparing Full Tableau and Revised Simplex Methods

Revised Simplex	Tableau Simplex
1. Storage: $\underbrace{B^{-1}}_{m\times m}, \underbrace{A}_{m\times n}, \bar{c}'$	$T = \left[\begin{array}{c\|c} -z_B & \bar{c}'^T \\ \hline \bar{x}_B & B^{-1}A \end{array}\right], \bar{c}'$ $\to (m+1)\times(n+1)$

B^{-1} is not sparse, but A often is (especially in large real-life LPs). And $B^{-1}A$ is not sparse. In practice n is often much larger than m.

2. pivots / operations

Revised Simplex	Tableau Simplex
Performed on B^{-1} ($m\times m$), followed by $p^T = \bar{c}_B^T B^{-1}$ and $\bar{c}'$. → Could compute c_j' one at a time.	Operations are performed on $T = \left[\begin{array}{c\|c} -z & \bar{c}'^T \\ \hline x_B & B^{-1}A \end{array}\right]$ $(m+1)\times(n+1)$

In summary, revised simplex is more efficient for large LPs.

The final project will ask you to compare your implementations of the revised simplex and the tableau simplex methods.