MATH464 – Lecture 21 (03/27/2018)

Today: * LP duality
* primal-dual relationships
* ~~weak duality~~ didn't get to it

Duality

Consider $\left\{ \begin{array}{ll} \min & x^2+y^2 \\ \text{s.t.} & x+y=1 \end{array} \right\}$

We know how to optimize functions of the form $f(x,y)$ in calculus. With that idea in mind, we try to include the constraint $x+y=1$ in a modified function.

Unconstrained minimization in Calculus: $\min\limits_{x,y} f(x,y)$

$\left.\dfrac{\partial f}{\partial x}=0,\ \dfrac{\partial f}{\partial y}=0\right\} \Rightarrow$ critical points

$\dfrac{\partial^2 f}{\partial x^2}>0 \quad \dfrac{\partial^2 f}{\partial y^2}>0$: local minimum

We write the **Lagrangian** $L(x,y,p) = x^2+y^2+p(1-x-y)$

price or penalty

p = price for not satisfying the constraint.

Minimize L: $\dfrac{\partial L}{\partial x}=0,\ \dfrac{\partial L}{\partial y}=0$ (p is a given constant)

$\Rightarrow \left.\begin{array}{l} 2x-p=0 \\ 2y-p=0 \end{array}\right\} \quad x=y=\frac{p}{2}.$ To satisfy $x+y=1$, we need $p=1$.

If we used $p(x+y-1)$ instead, we get $p=-1$.

$\Rightarrow x=y=\frac{1}{2}$ is the solution (critical point).

Indeed, $\dfrac{\partial^2 L}{\partial x^2}=\dfrac{\partial^2 L}{\partial y^2}=2$, and hence $x=y=\frac{1}{2}$ is a local minimum.

In fact, it is the global minimum here!

Consider $\left\{\begin{array}{l} \min \ \bar{c}^T\bar{x} \\ \text{s.t. } A\bar{x}=\bar{b} \\ \qquad \bar{x}\geq\bar{0} \end{array}\right\}$ (P) Assume $\bar{x}^*$ is an optimal solution (i.e., $A\bar{x}^*=\bar{b}$, $\bar{x}^*\geq\bar{0}$)

Convert (P) to a <u>relaxed</u> problem by choosing a vector of prices $\bar{p}$.
→ not unconstrained yet, as we need $\bar{x}\geq\bar{0}$

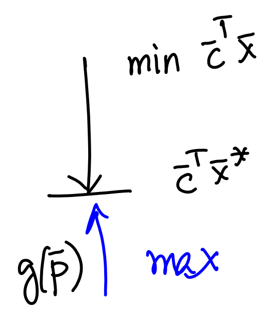
$$\min_{\bar{x}\geq\bar{0}} \left[\bar{c}^T\bar{x}+\bar{p}^T(\bar{b}-A\bar{x})\right] \text{ ——— } (P')$$

Let $g(\bar{p})$ be the optimal cost of (P') as a function of $\bar{p}$.
Notice $\bar{p}\in\mathbb{R}^m$. We get

$$\mathbf{g(\bar{p})\leq\bar{c}^T\bar{x}^*} \text{ for any } \bar{p}.$$

$$g(\bar{p})=\min_{\bar{x}\geq\bar{0}}\left[\bar{c}^T\bar{x}+\bar{p}^T(\bar{b}-A\bar{x})\right]\leq\bar{c}^T\bar{x}^*+\bar{p}^T\underbrace{(\bar{b}-A\bar{x}^*)}_{=\bar{0},\text{ as } \bar{x}^* \text{ is feasible}}$$
$$=\bar{c}^T\bar{x}^*$$

So, $g(\bar{p})$ is a lower bound for $\bar{c}^T\bar{x}$ (in (P)) for any choice of $\bar{p}$. Our goal is to find the largest lower bound, i.e., $\max g(\bar{p})$.



In general, we want largest lower bounds and smallest upper bounds.

For many optimization problems, we are often satisfied with good bounds on the value of objective function. Of course, for LP problems, we have algorithms that are guaranteed to terminate – either with an optimal solution, or with the certificate of infeasibility or unboundedness. But for harder classes of optimization problems such as integer programs (IPs), we are often satisfied with finding bounds. For example, we may find two solutions with values of objective function, say, 65 and 73, with the knowledge that the optimal objective function is in between the two values.

In this process of finding lower and upper bounds on the objective function, if we find lower bound = upper bound, we know that this is indeed the optimal value.

Coming back to the question of choosing an appropriate price vector $\bar{p}$, we want to find one such that $g(\bar{p})$ value is as large as possible – we are minimizing $\bar{c}^T\bar{x}$, and since $g(\bar{p})$ is a lower bound on $\bar{c}^T\bar{x}$ for any $\bar{p}$, we would like to push the lower bound as high as possible. In other words, we would like to maximize $g(\bar{p})$.

We will see later from the dual theorem that when $g(\bar{p}) = \bar{c}^T\bar{x}$, this value will be optimal for (P).

$$g(\bar{p}) = \min_{\bar{x} \geq \bar{0}} \left[\bar{c}^T \bar{x} + \bar{p}^T (\bar{b} - A\bar{x}) \right]$$

$$= \bar{p}^T \bar{b} + \min_{\bar{x} \geq \bar{0}} \left[\underbrace{\bar{c}^T \bar{x} - \bar{p}^T A \bar{x}} \right]$$

$$(\bar{c}^T - \bar{p}^T A)\bar{x} = \begin{cases} 0 & \text{if } \bar{c}^T - \bar{p}^T A \geq \bar{0} \\ -\infty & \text{if } \underbrace{\bar{c}^T - \bar{p}^T A \ngeq \bar{0}}_{\text{at least one entry } < 0} \end{cases}$$

We want to maximize $g(\bar{p})$, so we are interested only in cases where $\bar{c}^T - \bar{p}^T A \geq \bar{0}$, or $A^T \bar{p} \leq \bar{c}$. So we consider

$$\boxed{\begin{array}{ll} \max & \bar{p}^T \bar{b} \\ \text{s.t.} & \bar{p}^T A \leq \bar{c}^T \end{array}} \quad (D)$$

This is the **dual LP**.

could write also as $A^T \bar{p} \leq \bar{c}$.

(D) is the **dual linear program** to the original LP, which we refer to as the primal LP, (P). Here is the pair of LPs

$$(P) \quad \begin{array}{ll} \min & \bar{c}^T \bar{x} \\ \text{s.t.} & A\bar{x} = \bar{b} \\ & \bar{x} \geq 0 \end{array} \qquad\qquad \begin{array}{ll} \max & \bar{b}^T \bar{p} \\ \text{s.t.} & A^T \bar{p} \leq \bar{c} \end{array} \quad (D)$$

Given any LP we could write its corresponding dual LP in the same fashion. In particular, we need not necessarily start with an LP in standard form.

If, instead of $A\bar{x}=\bar{b}$ in (P), we had $A\bar{x} \geq \bar{b}$, we can transform it to $A\bar{x} - I\bar{s} = \bar{b}$ $\Rightarrow$ $[A \; -I]\begin{bmatrix}\bar{x}\\ \bar{s}\end{bmatrix} = \bar{b}$.

$\bar{s} \geq \bar{0}$

We get $\bar{p}^T [A \; -I] \leq [\bar{c}^T \; \bar{0}^T]$ → m-vector of zeros $\equiv \bar{s}$

$$\Rightarrow \begin{array}{l} \bar{p}^T A \leq \bar{c}^T \\ \bar{p} \geq \bar{0} \quad (-\bar{p}^T \leq \bar{0}^T) \end{array}$$

So $(P)\left\{\begin{array}{ll} \min & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x} \geq \bar{b} \\ & \bar{x} \geq \bar{0} \end{array}\right\}$ has the dual $\left\{\begin{array}{ll} \max & \bar{p}^T\bar{b} \\ \text{s.t.} & \bar{p}^T A \leq \bar{c}^T \\ & \bar{p} \geq \bar{0} \end{array}\right\}(D)$.

We could specify the dual LP for a general form LP:

$\bar{a}_i^T$: i^{th} row of A $\qquad M_1 \cup M_2 \cup M_3 = \{1, 2, \ldots, m\}$.

$\bar{A}_j$: j^{th} column of A $\qquad N_1 \cup N_2 \cup N_3 = \{1, 2, \ldots, n\}$.

$$(P) \quad \begin{array}{ll} \min & \bar{c}^T\bar{x} \\ \text{s.t.} & \bar{a}_i^T\bar{x} \geq b_i, \; i \in M_1 \\ & \bar{a}_i^T\bar{x} \leq b_i, \; i \in M_2 \\ & \bar{a}_i^T\bar{x} = b_i, \; i \in M_3 \\ & x_j \geq 0, \; j \in N_1 \\ & x_j \leq 0, \; j \in N_2 \\ & x_j \text{ urs}, \; j \in N_3 \end{array}$$

$$\begin{array}{ll} \max & \bar{p}^T\bar{b} \\ \text{s.t.} & p_i \geq 0, \; i \in M_1 \\ & p_i \leq 0, \; i \in M_2 \\ & p_i \text{ urs}, \; i \in M_3 \\ & \bar{p}^T A_j \leq c_j, \; j \in N_1 \\ & \bar{p}^T A_j \geq c_j, \; j \in N_2 \\ & \bar{p}^T A_j = c_j, \; j \in N_3 \end{array} \quad (D)$$

Table of primal-dual relationships

We could specify all these correspondences in a table as follows.

Primal	min	max	Dual
variables	≥ 0 ≤ 0 urs	$\leq$ $\geq$ $=$	constraints
constraints	$\geq$ $\leq$ $=$	≥ 0 ≤ 0 urs	variables

normal constraints and vars

min LP – $\geq$ constraint is normal

min cost s.t. meet demand

max-LP – $\leq$ constraint is normal

max revenue s.t. limited resources

≥ 0 variables are always normal

The table of primal dual relationships could be presented in several equivalent forms – you need not memorize any one form! Here are the general rules.

→ variable in the primal LP $\Longleftrightarrow$ constraint in dual LP.

→ normal (opposite to normal) variable in primal LP $\Longleftrightarrow$ normal (opposite to normal) constraint in dual LP.

→ urs variable in primal LP $\Longleftrightarrow$ $=$ constraint in dual LP.

normal vars : ≥ 0

normal constraints : → $\leq$ for **max-LP** {max revenue s.t. upper bound on raw materials}
→ $\geq$ for **min-LP** {min cost s.t. make at least so many # products, i.e., meet demand.}

Let's consider an example, and apply these relationships directly:

(P)

$$\begin{array}{lll} \max & 5x_1+4x_2-3x_3 & \\ \text{s.t.} & x_1 \quad -5x_3 \geq 4 & y_1 \leq 0 \\ & 3x_1+x_2+2x_3 \leq 5 & y_2 \geq 0 \\ & x_1 \geq 0,\ x_2 \text{ urs},\ x_3 \geq 0 & \\ & \geq \quad = \quad \geq & \end{array}$$

(D)

$$\begin{array}{ll} \min & 4y_1+5y_2 \\ \text{s.t.} & y_1+3y_2 \geq 5 \\ & y_2 = 4 \\ & -5y_1+2y_2 \geq -3 \\ & y_1 \leq 0,\ y_2 \geq 0 \end{array}$$

If we take the dual of the dual LP, we get back the primal LP:

(D)

$$\begin{array}{lll} \min & 4y_1+5y_2 & \\ \text{s.t.} & y_1+3y_2 \geq 5 & u_1 \geq 0 \\ & y_2 = 4 & u_2 \text{ urs} \\ & -5y_1+2y_2 \geq -3 & u_3 \geq 0 \\ & y_1 \leq 0,\ y_2 \geq 0 & \\ & \geq \quad \leq & \end{array}$$

(D')

$$\begin{array}{ll} \max & 5u_1+4u_2-3u_3 \\ \text{s.t.} & u_1 \quad -5u_3 \geq 4 \\ & 3u_1+u_2+2u_3 \leq 5 \\ & u_1 \geq 0,\ u_2 \text{ urs},\ u_3 \geq 0 \end{array}$$

(D')

$$\begin{array}{ll} \max & 5u_1+4u_2-3u_3 \\ \text{s.t.} & u_1 \quad -5u_3 \geq 4 \\ & 3u_1+u_2+2u_3 \leq 5 \\ & u_1 \geq 0,\ u_2 \text{ urs},\ u_3 \geq 0 \end{array}$$

(P)

$$\begin{array}{ll} \max & 5x_1+4x_2-3x_3 \\ \text{s.t.} & x_1 \quad -5x_3 \geq 4 \\ & 3x_1+x_2+2x_3 \leq 5 \\ & x_1 \geq 0,\ x_2 \text{ urs},\ x_3 \geq 0 \end{array}$$

Notice (D') is equivalent to (P)!