MATH 464 – Lecture 22 (03/29/2018)

Today:
* duality in matrix form
* weak duality
* strong duality

Table of primal-dual relationships

Primal	max	min	Dual
constraints	$\leq$	≥ 0	variables
	$\geq$	≤ 0	
	$=$	urs	
variables	≥ 0	$\geq$	constraints
	≤ 0	$\leq$	
	urs	$=$	

Duality in Matrix Form

(P)
$$\begin{aligned} \min\ & \bar{c}^T \bar{x} \\ \text{s.t. } & A\bar{x} \leq \bar{b} \quad \bar{p} \leq \bar{0} \\ & \quad = \end{aligned}$$

(D)
$$\begin{aligned} \max\ & \bar{b}^T \bar{p} \\ \text{s.t. } & A^T \bar{p} = \bar{c} \\ & \bar{p} \leq \bar{0} \end{aligned}$$

(P)
$$\begin{aligned} \max\ & \bar{c}^T \bar{x} + \bar{d}^T \bar{y} \\ \text{s.t. } & A\bar{x} + B\bar{y} = \bar{f} \quad \bar{p} \text{ urs} \\ & \bar{x} \geq \bar{0}, \ \bar{y} \leq \bar{0} \\ & \quad \geq \quad\quad \leq \end{aligned}$$

(D)
$$\begin{aligned} \min\ & \bar{f}^T \bar{p} \\ \text{s.t. } & A^T \bar{p} \geq \bar{c} \\ & B^T \bar{p} \leq \bar{d} \\ & \bar{p} \text{ urs} \end{aligned}$$

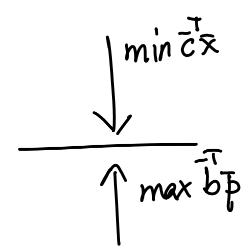
Weak Duality (BT-ILO Theorem 4.3)

Consider

$$(P)\quad \begin{array}{ll} \min & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}=\bar{b} \quad \bar{p} \\ & \bar{x} \geqslant \bar{0} \\ & \quad \leqslant \end{array} \qquad \text{and} \qquad \begin{array}{ll} \max & \bar{b}^T\bar{p} \\ \text{s.t.} & A^T\bar{p} \leqslant \bar{c} \\ & \bar{p} \text{ urs} \end{array} \quad (D).$$

Let $\bar{x}$ be feasible for (P) and $\bar{p}$ be feasible for (D). Then $\bar{\mathbf{b}}^T\bar{\mathbf{p}} \leqslant \bar{\mathbf{c}}^T\bar{\mathbf{x}}$.

Recall the intuitive picture of pulling $\bar{c}^T\bar{x}$ down from above, and pushing $\bar{b}^T\bar{p}$ up from below.



The values of $\bar{c}^T\bar{x}$ for any feasible $\bar{x}$ always lie above the values of $\bar{b}^T\bar{p}$ for any feasible $\bar{p}$.

Proof $\bar{p}$ is feasible for (D) gives

$$\left(A^T\bar{p} \leqslant \bar{c}\right)^T \qquad \text{take transpose on both sides}$$

$$\Longrightarrow \quad \bar{p}^TA \leqslant \bar{c}^T$$

$\bar{x}$ is feasible for (P) gives $\bar{x} \geqslant \bar{0}$, $A\bar{x}=\bar{b}$.

$$\left(\bar{p}^TA \leqslant \bar{c}^T\right)\bar{x} \qquad \text{multiply by } \bar{x} \text{ on the right (on both sides)}$$

$$\Longrightarrow \quad \bar{p}^T\underbrace{A\bar{x}}_{\bar{b}} \leqslant \bar{c}^T\bar{x} \qquad \text{stays} \leqslant \text{ as } \bar{x} \geqslant \bar{0}$$

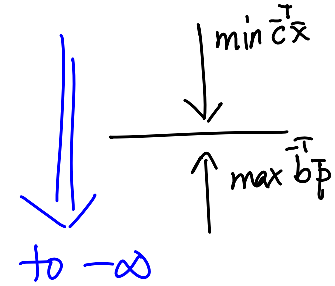
$$\Longrightarrow \quad \bar{p}^T\bar{b} \leqslant \bar{c}^T\bar{x}.$$

$\square$

We get the following two corollaries.

Corollary 4.1 If the primal optimal cost is $-\infty$, then the dual is infeasible. Similarly, if the dual cost is $+\infty$, then the primal is infeasible.

Note that the primal (dual) is unbounded when its optimal cost is $-\infty$ $(+\infty)$.



More generally, if an LP is unbounded, its dual LP is infeasible. Going back to the intuitive picture, if we can keep pulling down $\bar{c}^T\bar{x}$ without limit, then there could be no finite lower bound given by any $\bar{b}^T\bar{p}$, i.e., there is no feasible $\bar{p}$ for dual (D).

Corollary 4.2 If $\bar{x}, \bar{p}$ are feasible for (P) and (D), respectively, and $\bar{b}^T\bar{p} = \bar{c}^T\bar{x}$, then $\bar{x}$ and $\bar{p}$ are optimal for (P) and (D), respectively.

Proof $\bar{b}^T\bar{p} = \bar{c}^T\bar{x}$. Weak duality gives that $\bar{b}^T\bar{p} \le \bar{c}^T\bar{y}$ for any $\bar{y}$ that is a feasible solution for (P). Hence we get $\bar{c}^T\bar{x} \le \bar{c}^T\bar{y}$ for any $\bar{y}$ that is a feasible solution for (P), i.e., $\bar{x}$ is optimal.

A similar argument holds for (D).

Strong Duality (BT-ILO Theorem 4.4)

If an LP has an optimal solution, then so does its dual, and the optimal costs are equal.

Proof Let $\bar{x}$ be optimal for (P) $\{\min \bar{c}^T\bar{x} \mid A\bar{x}=\bar{b}, \bar{x} \geq \bar{0}\}$. Let B be the corresponding optimal basis matrix. Then $\bar{x}_B = B^{-1}\bar{b}$.

The reduced costs for (P) must be non-negative, from the optimality conditions for (P).

$$\Rightarrow \quad \bar{c}'^T = \bar{c}^T - \underbrace{\bar{c}_B^T B^{-1}}_{\bar{p}^T} A \geq \bar{0}^T$$

Define $\bar{p}^T = \bar{c}_B^T B^{-1} \Longrightarrow (\bar{c}^T - \bar{p}^T A \geq \bar{0}^T)^T$

$\Longrightarrow \quad A^T\bar{p} \leq \bar{c}$, i.e., $\bar{p}$ is feasible for (D).

Also, $\bar{b}^T\bar{p} = \bar{p}^T\bar{b} = \bar{c}_B^T B^{-1}\bar{b} = \bar{c}_B^T\bar{x}_B = \bar{c}^T\bar{x}$.

Hence, by weak duality (Corollary 4.2), $\bar{p}$ is optimal for (D). $\square$

Possibilities for (P) and (D)

		Dual		
		finite optimum	unbounded	infeasible
	finite optimum	✓	X	X
Primal	unbounded	X	X	✓
	infeasible	X	✓	✓

An 'X' means that combination is not possible. For instance, we cannot have an infeasible primal LP for which the dual LP has a finite optimum.

Both (P) and (D) could be infeasible:

$$(P) \quad \begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 = 2 \quad p_1 \\ & 3x_1 + 3x_2 = 4 \quad p_2 \\ & \quad = \qquad = \end{array}$$

$$\begin{array}{ll} \max & 2p_1 + 4p_2 \\ \text{s.t.} & p_1 + 3p_2 = 1 \\ & p_1 + 3p_2 = 2 \end{array} \quad (D)$$