MATH464 – Lecture 23 (04/04/2023)

Today:
* complementary slackness conditions
* interpretation of duality
* economic interpretation of dual

Duality:

(P) $\min \bar{c}^T\bar{x}$ s.t. $A\bar{x}=\bar{b}$ $\bar{p}$, $\bar{x}\geq\bar{0}$

$\max \bar{b}^T\bar{p}$ s.t. $A^T\bar{p}\leq\bar{c}$ (D)

Complementary Slackness Conditions (CSCs) (BT-ILO Theorem 4.5)

Let $\bar{x},\bar{p}$ be feasible for (P) (min LP in general form) and (D), respectively. They are optimal iff

$$p_i(\bar{a}_i^T\bar{x}-b_i)=0 \quad \forall i \ (i=1,\dots,m)$$
$$(c_j-\bar{p}^TA_j)x_j=0 \quad \forall j \ (j=1,\dots,n)$$ hold.

Proof Let $u_i=p_i(\bar{a}_i^T\bar{x}-b_i)$ and $v_j=(c_j-\bar{p}^TA_j)x_j$.

By the definition of the dual LP, we get $u_i\geq 0\ \forall i$, as explained below.

If constraint i is $\bar{a}_i^T\bar{x}\geq b_i$ then $p_i\geq 0 \Rightarrow u_i\geq 0$, and
if constraint i is $\bar{a}_i^T\bar{x}\leq b_i$ then $p_i\leq 0 \Rightarrow u_i\geq 0$.

But if constraint i is $\bar{a}_i^T\bar{x}=b_i$, then $u_i=0$ (p_i is urs, but it does not matter).
Hence $u_i\geq 0\ \forall i$ holds. Similarly, $v_j\geq 0\ \forall j$.

So, $$\sum_{i=1}^m u_i=\sum_{i=1}^m p_i(\bar{a}_i^T\bar{x}-b_i)=\bar{p}^T(A\bar{x}-\bar{b})=\bar{p}^TA\bar{x}-\bar{p}^T\bar{b},$$ and
$$\sum_{j=1}^n v_j=\bar{c}^T\bar{x}-\bar{p}^TA\bar{x}.$$

$$\Rightarrow \sum_{i=1}^m u_i+\sum_{j=1}^n v_j=\bar{c}^T\bar{x}-\bar{p}^T\bar{b}=0$$ iff $\bar{x},\bar{p}$ are optimal for (P) and (D), by strong duality.

Hence we have $u_i \geq 0 \ \forall i$; $v_j \geq 0 \ \forall j$, $\sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j = 0$ at optimality, which means $u_i = 0 \ \forall i$ and $v_j = 0 \ \forall j$ at optimality (iff). $\square$

If a constraint is **not** active, its corresponding dual variable is zero at optimality. Similarly, if a variable is **nonzero** at optimality, its corresponding dual constraint is active (at optimality).

Caution: The reverse implication might not hold, i.e., if a constraint is active, its dual variable could be zero or nonzero.

<u>An Example</u>: Consider the following pair of primal-dual LPs:

(P)
$$
\begin{aligned}
\max \quad & 3x_1 + 4x_2 + x_3 + 5x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 + 2x_4 \leq 5 \qquad p_1 \geq 0 \\
& 2x_1 + 3x_2 + x_3 + 3x_4 \leq 8 \qquad p_2 \geq 0 \\
& x_j \geq 0 \ \forall j \\
& \quad \geq
\end{aligned}
$$

(D)
$$
\begin{aligned}
\min \quad & 5p_1 + 8p_2 \\
\text{s.t.} \quad & p_1 + 2p_2 \geq 3 \quad \text{—(1)} \\
& 2p_1 + 3p_2 \geq 4 \quad \text{—(2)} \\
& p_1 + p_2 \geq 1 \quad \text{—(3)} \\
& 2p_1 + 3p_2 \geq 5 \quad \text{—(4)} \\
& p_1, p_2 \geq 0
\end{aligned}
$$

It is given that constraints (1) and (4) are active in an optimal solution of (D). Using that fact, and CSCs, solve (P).

(1) and (4) are active $\Rightarrow$ $\left.\begin{aligned} p_1 + 2p_2 &= 3 \\ 2p_1 + 3p_2 &= 5 \end{aligned}\right\}$ $p_1 = p_2 = 1$.

by CSCs

At $p_1 = 1, p_2 = 1$, Constraints (2) and (3) are not active, $x_2 = x_3 = 0$ in (P).

Also, $p_1 > 0, p_2 > 0 \Rightarrow$ constraints in (P) are active.

$\Rightarrow$ In (P) at optimality, $\left.\begin{aligned} x_1 + 2x_4 &= 5 \\ 2x_1 + 3x_4 &= 8 \end{aligned}\right\}$ $\Rightarrow x_1 = 1, x_4 = 2$ $(x_2 = x_3 = 0)$

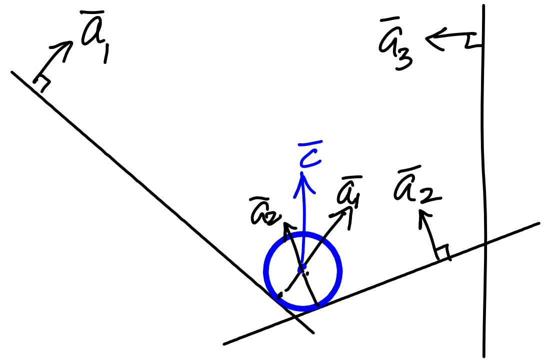
Indeed $\bar{c}^T \bar{x} = \bar{p}^T \bar{b} = 13$, and hence we have solved (P) and (D).

An intuitive, mechanical analogy of duality

Consider the primal-dual pair of LPs:

$$(P)\quad \min\ \bar{c}^T\bar{x} \quad \text{s.t. } A\bar{x} \geq \bar{b}$$

$$\max\ \bar{p}^T\bar{b} \quad \text{s.t. } \bar{p}^TA = \bar{c}^T,\ \bar{p} \geq \bar{0} \quad (D)$$



Consider a bowl determined by the constraints $\bar{a}_i^T\bar{x} \geq b_i$, and a ball that is being pulled down by gravity ($\equiv \min\ \bar{c}^T\bar{x}$). The ball comes to rest at the lowest point in the bowl.

At equilibrium, the force provided by the walls touching the ball balance gravity (i.e., its weight). The forces excerted by the walls act perpendicular to the walls. Let p_i be the force applied by wall i. Hence we have $\sum_{i=1}^{m} p_i\bar{a}_i = \bar{c}$ for some non-negative magnitudes of forces p_i.

These equilibrium p_i's are an optimal solution to (D). Also, $p_i=0$ if the wall i (represented by $\bar{a}_i$) does not touch the ball at equilibrium. So, if $\bar{a}_i^T\bar{x} > b_i$, $p_i = 0$; which is the CSC for that constraint-dual variable pair. Hence we get that

$$p_i(\bar{a}_i^T\bar{x} - b_i) = 0 \quad \forall i.$$

$$\Rightarrow \bar{p}^T\bar{b} = \sum_i p_i b_i = \sum_i p_i \bar{a}_i^T\bar{x} = \underbrace{\bar{p}^TA}_{\bar{c}^T\text{, from (D)}}\bar{x} = \bar{c}^T\bar{x}, \text{ confirming}$$

that $\bar{x}$ and $\bar{p}$ are optimal for (P) and (D), respectively.

Economic Interpretation

Exercise 1.15 A company produces two kinds of products. A product of the first type requires 1/4 hours of assembly labor, 1/8 hours of testing, and \$1.2 worth of raw materials. A product of the second type requires 1/3 hours of assembly, 1/3 hours of testing, and \$0.9 worth of raw materials. Given the current personnel of the company, there can be at most 90 hours of assembly labor and 80 hours of testing, each day. Products of the first and second type have a market value of \$9 and \$8, respectively.

(a) Formulate a linear programming problem that can be used to maximize the daily profit of the company.

Here is the LP:

3. Let x_j = # units of product j made, for $j = 1, 2$. Here is the LP.

$$\begin{array}{llcll}
\max & (9-1.2)x_1 + (8-0.9)x_2 & & & \text{(daily profit)} \\
\text{s.t.} & (1/4)\,x_1 + (1/3)\,x_2 & \leq & 90 & \text{(assembly hours)} \quad p_1 \geq 0 \\
& (1/8)\,x_1 + (1/3)\,x_2 & \leq & 80 & \text{(testing hours)} \quad p_2 \geq 0 \\
& x_1, x_2 & \geq & 0 & \text{(non-negativity)}
\end{array}$$

$\geq \; \geq$

And here is the dual LP:

$$\begin{array}{lll}
\min & 90p_1 + 80p_2 & (\;?\;) \\
\text{s.t.} & \left(\tfrac{1}{4}\right)p_1 + \left(\tfrac{1}{8}\right)p_2 \geq 7.8 & (\;?\;) \\
& \left(\tfrac{1}{3}\right)p_1 + \left(\tfrac{1}{3}\right)p_2 \geq 7.1 & (\;?\;) \\
& p_1, p_2 \geq 0 & \text{(non-neg)}
\end{array}$$

How can we interpret the objective function and constraints? We consider a scenario where a competing firm wants to buy out (the operations of) this company.

Let a competing firm consider buying out the company. It will have to pay for the hours of assembly and testing. Let p_1 and p_2 be the price per hour of assembly and testing, respectively, that this firm is offering. Since they are prices, they must be nonnegative.

The firm's offer must be attractive to the company. If the company has $\frac{1}{4}$ hr of assembly and $\frac{1}{8}$ hr of testing available, it can make one unit of product 1, which gives a profit of \$7.8. Hence, p_1 and p_2 should be such that $\left(\frac{1}{4}\right)p_1 + \left(\frac{1}{8}\right)p_2 \geq 7.8$, which is the first constraint. The second constraint is interpreted in a similar fashion for product 2.

The firm would like to minimize the total cost of this purchase, i.e., minimize $90p_1 + 80p_2$, which is the objective function.

We could interpret the (optimal) dual variables as "shadow prices" — more on this idea in the next lecture.