MATH464 – Lecture 24 (04/06/2023)

Today:
* More on AMPL
* Dual variables as shadow prices
* Farkas' Lemma

Hw9 is now due on Monday, April 10.

AMPL session: **read** command to read in parameters from text files. Display optimal dual variables using names of primal constraints.

See AMPL session for details:

https://www.math.wsu.edu/faculty/bkrishna/FilesMath464/S23/Software/AMPL/Lec1_01102023_Session.txt

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Back to Economic Interpretation of Dual LP

The optimal solution values for p_1 and p_2 could be interpreted as **shadow prices** for assembly and testing hours.

The optimal solution for (D) is $p_1 = 31.2$, $p_2 = 0$.

At optimality in (P), we are using all of the 90 assembly hours available, but only 45 out of the 80 testing hours available. The company could consider buying more assembly hours. The company would be willing to pay up to $p_1 = \$31.2$ for each extra assembly hour.

But the amount the company would pay for an extra testing hour is $p_2 = 0$, as it is currently not using all testing hours available.

The second result is also a direct interpretation of complementary slackness. Since (testing hours) constraint is not active, its dual variable is zero at optimality. Since we are not using all of the 80 hrs of testing available, we will not pay anything for more testing hours.

Farkas' Lemma

Consider $\left\{ \begin{array}{l} A\bar{x}=\bar{b} \\ \bar{x} \geq \bar{0} \end{array} \right\}$. Does this system have a solution?

To answer this question completely, we need to justify our YES/NO response. If the answer is YES, we could specify an $\bar{x}$ that satisfies the system. Indeed, we could easily check whether $\bar{x}$ satisfies $A\bar{x}=\bar{b}$ and $\bar{x} \geq \bar{0}$.

What if our response is NO? How do we justify this response?
Suppose there exists a $\bar{p}$ such that $\bar{p}^T A \geq \bar{0}, \bar{p}^T\bar{b} < 0$.
Then since $\bar{x} \geq \bar{0}$, for any potential solution $\bar{x}$, we get $\bar{p}^T \underbrace{A\bar{x}}_{\bar{b}} \geq 0$, which conflicts with $\bar{p}^T\bar{b} < 0$. Hence, such a $\bar{p}$ could be provided as a certificate of infeasibility for the original system. Farkas' lemma provides such a result – either the original system is feasible or an alternative system is feasible that certifies the infeasibility of the original system.

Farkas' lemma is a classic example of such "systems of alternatives" type result, which are seen in many subfields of mathematics. We could prove Farkas' lemma from first principles, but the proof would be harder than if we use LP duality.

(BT-ILO Theorem 4.6) Let $A \in \mathbb{R}^{m \times n}$, $\bar{b} \in \mathbb{R}^m$. Then exactly one of the following two statements hold.

(a) There exists some $\bar{x} \geqslant \bar{0}$ such that $A\bar{x} = \bar{b}$.

(b) There exists some $\bar{p}$ such that $\bar{p}^T A \geqslant \bar{0}$ and $\bar{p}^T \bar{b} < 0$.

Proof ($\Rightarrow$) If (a) holds, there exists $\bar{x} \geqslant \bar{0}$ such that $A\bar{x} = \bar{b}$.

$$\Rightarrow \bar{p}^T A \bar{x} = \bar{p}^T \bar{b}.$$

If $\bar{p}^T A \geqslant \bar{0}$ then $\bar{p}^T \bar{b} = \bar{p}^T (A\bar{x}) = (\bar{p}^T A)\bar{x} \geqslant 0$, as $\bar{x} \geqslant \bar{0}$.

$\Rightarrow$ (b) cannot hold.

If (b) holds, there exists $\bar{p}$ such that $\bar{p}^T A \geqslant \bar{0}$ and $\bar{p}^T \bar{b} < 0$.

For any $\bar{x} \geqslant \bar{0}$, $(\bar{p}^T A)\bar{x} \geqslant 0$, but $\bar{p}^T \bar{b} < 0$ means $A\bar{x} \neq \bar{b}$.

So, (a) cannot hold.

This is the "easy" direction of the proof. For showing the reverse direction ($\Leftarrow$), we will use LP duality.

($\Leftarrow$) If (a) does not hold, we want to show (b) holds.

Let $\nexists\, \bar{x} \geqslant 0$ s.t. $A\bar{x} = \bar{b}$ ((a) does not hold).

Consider the following primal-dual pair of LPs:

$$(P) \quad \begin{array}{ll} \max & \bar{0}^T \bar{x} \\ \text{s.t.} & A\bar{x} = \bar{b} \quad \bar{p} \\ & \bar{x} \geqslant \bar{0} \\ & \geqslant \end{array} \qquad\qquad \begin{array}{ll} \min & \bar{p}^T \bar{b} \\ \text{s.t.} & \bar{p}^T A \geqslant \bar{0}^T \\ & \bar{p} \text{ urs} \end{array} \quad (D)$$

(P) is infeasible here. Hence (D) is infeasible or unbounded.
But $\bar{p}=\bar{0}$ is feasible for (D), and hence it is unbounded, i.e., $\bar{p}^T\bar{b} \to -\infty$.
Hence some $\bar{p}$ exists that is feasible for (D), and $\bar{p}^T\bar{b} < 0$. Hence, (b) holds.

If (b) does not hold, then $\bar{p}^T\bar{b} \geqslant 0 \;\forall\; \bar{p}$ such that $\bar{p}^T A \geqslant \bar{0}$.
This condition means (D) is feasible and is not unbounded, i.e., it must have an optimal solution. This result follows from the observation that $\bar{p}^T\bar{b}$ cannot be decreased without limit.
By duality, (P) must have an optimal solution as well.
Hence (P) has a feasible solution, i.e., (a) holds. $\square$

Note how we came up with the primal-dual pair of LPs.
Since the rhs of the system in (b) is $\bar{0}$ (from $\bar{p}^T A \geqslant \bar{0}$), that is precisely the objective function of the primal LP (P).
Also, since the constraints in (P) are equations ($A\bar{x}=\bar{b}$), the dual variables $\bar{p}$ are urs.