MATH 464 – Lecture 25 (04/11/2023)

Today:
* illustration of Farkas' lemma
* asset pricing (application of Farkas' lemma).
* optimal dual prices as marginal costs.

On AMPL: For the inventory problem, best to set param n as the # months so that you can use 1..n or 0..n-1 or 0..n as the required sets to write, e.g., the balance constraint in one line:

s[i-1] + x[i] = Demand[i] + s[i]; for i = 1,..,n.

You could work with a set Months := m1 m2 ... m12; but then you can't directly index previous/next month: AMPL will **not** take m2+1 as m3!

<u>Farkas' lemma</u>

(a) $\exists\, \bar{x} \geqslant \bar{0}$ s.t. $A\bar{x} = \bar{b}$

(b) $\exists\, \bar{p}$ s.t. $\bar{p}^T A \geqslant \bar{0}$ and $\bar{p}^T \bar{b} < 0$.

<u>Illustration in 2D</u>

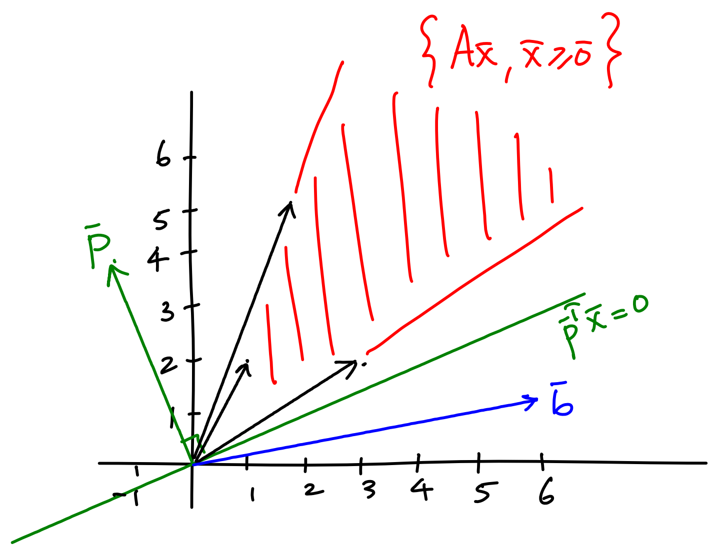
$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 5 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$.

Does there exist $\bar{x} \geqslant \bar{0}$ s.t. $A\bar{x} = \bar{b}$?

complicating part.

No, as seen in the picture!

$\bar{b}$ is not in the cone $\{A\bar{x} \mid \bar{x} \geqslant \bar{0}\}$.



For $\bar{p} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, we get $\bar{p}^T A = [-1 \;\; 4]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 5 \end{bmatrix} = [7 \;\; 5 \;\; 18] \geqslant \bar{0}$

and $\bar{p}^T \bar{b} = [-1 \;\; 4]\begin{bmatrix} 6 \\ 1 \end{bmatrix} = -2$.

We see that $\{A\bar{x} \mid \bar{x} \geqslant \bar{0}\}$ and $\bar{b}$ lie on opposite sides of $\{\bar{x} \mid \bar{p}^T\bar{x} = 0\}$. Hence the vector $\bar{p}$ gives a certificate of infeasibility of the original system. Note that there could be multiple such $\bar{p}$ vectors possible here – any solution to the alternative system in (b) will do.

In this example, note that the $\bar{x} \geq \bar{0}$ restriction is indeed the "complicating" constraint. If we wanted to check if $A\bar{x}=\bar{b}$ (for any $\bar{x}$), then answer would have been YES, since the columns of A span $\mathbb{R}^2$ here. But we could present Farkas' lemma in an equivalent form without $\bar{x} \geq \bar{0}$ in alternative (a).

Alternative forms of Farkas' lemma

Exactly one of the following two alternatives hold:

(a) $\exists\, \bar{x}$ s.t. $A\bar{x} = \bar{b}$.

(b) $\exists\, \bar{p}$ s.t. $\bar{p}^T A = \bar{0}$, $\bar{p}^T \bar{b} = -1$. $\longrightarrow$ similar to $[0\ 0 \dots 0 \mid \blacksquare]$ where $\blacksquare \neq 0$, which indicates $A\bar{x}=\bar{b}$ is inconsistent.

Idea: If (b) holds, we can derive (by taking linear combinations of rows) $0=-1$ from $A\bar{x}=\bar{b}$, and hence (a) cannot hold.

Proof (part of).

$(\Leftarrow)$ If (a) does not hold, (b) holds.

$$(P)\quad \begin{array}{ll} \max & \bar{0}^T\bar{x} \\ \text{s.t.} & A\bar{x}=\bar{b} \quad \bar{p} \end{array} \qquad\qquad \begin{array}{ll} \min & \bar{p}^T\bar{b} \\ \text{s.t.} & \bar{p}^T A = \bar{0}^T \end{array} \quad (D)$$

(a) does not hold means (P) is infeasible. So (D) is either infeasible or unbounded. But $\bar{p}=\bar{0}$ is feasible for (D), so (D) is unbounded. Hence $\exists\, \bar{p}$ s.t. $\bar{p}^T A = \bar{0}^T$ and $\bar{p}^T\bar{b} < 0$. We can scale $\bar{p}$ such that $\bar{p}^T\bar{b} = -1$ holds. Hence (b) holds.

Application of Farkas' Lemma – Asset Pricing

There are n assets, and after a period of investment there are m possible states (or scenarios). Let r_{ij} be the return from asset j under state/scenario i. Hence if you invest \$1 in asset j, and the state turns out to be i, you expect to receive $\$r_{ij}$.

Assumption 1 r_{ij} is same for both $x_j \geq 0$ and $x_j < 0$.

($x_j < 0$ means you're in a "short" position — you sell $|x_j|$ at the start, with the promise to buy back at the end at rate r_{ij} if state is i. The payoff of $r_{ij}x_j < 0$ here, meaning you have to pay $r_{ij}|x_j|$).

Assumption 1 might not hold in general, but that's not too critical for our illustration here.

Let $R = [r_{ij}] \in \mathbb{R}^{m \times n}$, and $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be the amount of assets at start ($x_j \geq 0$ and $x_j < 0$ are both possible).

Let $\bar{w}$ represent the wealth at the end of the period. Then we write $\bar{w} = R\bar{x}$. Let $\bar{p}$ be the prices at the start for each asset. Then $\bar{p}^T\bar{x}$ is the total cost of the portfolio.

An **arbitrage** is a price such that $R\bar{x} \geq \bar{0}$ but $\bar{p}^T\bar{x} < 0$, i.e., we get a non-negative return for a negative price.

Assumption 2 Prices will equilibriate to avoid arbitrages.

"No arbitrages" means "if $R\bar{x} \geq \bar{0}$ then $\bar{p}^T\bar{x} \geq \bar{0}$".

(BT-ILO Theorem 4.8) There exist no arbitrages iff there exists

$$\bar{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \geq \bar{0} \text{ such that } \bar{p} = R^T\bar{q},\ \bar{q} \geq \bar{0}.$$

Intuitively, this theorem says that there will be no arbitrages iff the prices are some non-negative linear combination of the return rates $R = [r_{ij}]$. This result is usually presented independent of any connections to linear programming. But we can prove it by a direct application of Farkas' lemma.

Proof Follows from Farkas' lemma

(a) $\exists\, \bar{x} \geq \bar{0}$ s.t. $A\bar{x} = \bar{b}$.
(b) $\exists\, \bar{p}$ s.t. $\bar{p}^T A \geq \bar{0}$, $\bar{p}^T\bar{b} < 0$.

We construct one alternative to represent arbitrages. The other alternative then presents the criterion given in the theorem for existence of $\bar{q}$.

(a) $\bar{R}^T\bar{q} = \bar{p}$, $\bar{q} \geq \bar{0}$

(b) $R\bar{x} \geq \bar{0}$, $\bar{p}^T\bar{x} < 0$ (arbitrage)

Optimal dual variables as marginal costs

$$
(P)\quad \begin{array}{ll} \min & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}=\bar{b} \\ & \bar{x}\geqslant\bar{0} \end{array}
$$

Assume (P) has optimal solution $\bar{x}^*$ that is non-degenerate, and let B be the corresponding basis matrix.

Then $\bar{x}_B = B^{-1}\bar{b} > \bar{0}$. Optimal cost is $\bar{c}_B^T\bar{x}_B$.

Also, the reduced costs: $\bar{c}'^T = \bar{c}^T - \bar{c}_B^T B^{-1}A \geqslant \bar{0}$ (optimality conditions).

Consider changing $\bar{b}$ to $\bar{b}+\bar{d}$ (rhs vector). Since $\bar{x}_B > \bar{0}$, for $\bar{d}$ small enough, the new optimal solution $\bar{x}_B = B^{-1}(\bar{b}+\bar{d}) \geqslant \bar{0}$. And hence the new optimal cost is $\bar{c}_B^T B^{-1}(\bar{b}+\bar{d}) = \underbrace{\bar{c}_B^T B^{-1}\bar{b}}_{\text{original cost}} + \underbrace{\bar{c}_B^T B^{-1}}_{\bar{p}^T}\bar{d}$.

Hence we get new optimal cost = old optimal cost + $\bar{p}^T\bar{d}$. Thus $\bar{p}$ can be interpreted as the marginal cost (or shadow price) for increasing each rhs entry by 1 unit. In particular, p_i is the "shadow price for "resource" i, i.e., the price to pay for increasing b_i to b_i+1.

Dual Simplex Method

Caution! This is **NOT** applying simplex method to the dual LP!

Tableau for the primal simplex method:

$$(P)\quad \begin{aligned} \min\ & \bar{c}^T\bar{x} \\ \text{s.t. } & A\bar{x} = \bar{b} \\ & \bar{x} \geq \bar{0} \end{aligned} \qquad\qquad \begin{aligned} \max\ & \bar{p}^T\bar{b} \\ \text{s.t. } & \bar{p}^T A \leq \bar{c}^T \end{aligned} \quad (D)$$

$\bar{c}' \geq 0$ (maintain)

$-\bar{c}_B^T\bar{x}_B$	$\bar{c}^T - \bar{c}_B^T B^{-1}A$
$B^{-1}\bar{b}$	$B^{-1}A$

≥ 0

allow < 0
strive to make them ≥ 0.

more in the next lecture...