MATH 464 – Lecture 26 (04/13/2023)

Today: * dual simplex method
* proof exercises from HW7

Dual Simplex Method

Tableau for the primal simplex method:

$-\bar{c}_B^T\bar{x}_B$	$\underbrace{\bar{c}^T-\bar{c}_B^TB^{-1}A}_{\bar{c}'^T}$
$B^{-1}\bar{b}$	$B^{-1}A$

$$(P)\quad \begin{array}{ll}\min & \bar{c}^T\bar{x}\\ \text{s.t.} & A\bar{x}=\bar{b}\\ & \bar{x}\geqslant\bar{0}\end{array} \qquad\qquad \begin{array}{ll}\max & \bar{p}^T\bar{b}\\ \text{s.t.} & \bar{p}^TA\leq\bar{c}^T\end{array}\quad (D)$$

Optimality Conditions $\quad \bar{x}_B=B^{-1}\bar{b}\geqslant\bar{0}$ (feasibility) and

(optimality) $\quad \bar{c}'^T=\bar{c}^T-\underbrace{\bar{c}_B^TB^{-1}}_{\bar{p}^T}A\geqslant\bar{0}^T \implies \bar{p}^TA\leq\bar{c}^T$

So, optimality for (P) $\iff$ feasibility for (D)

$$\text{Primal cost}=\bar{c}_B^T\bar{x}_B=\underbrace{\bar{c}_B^TB^{-1}}_{\bar{p}^T}\bar{b}=\bar{p}^T\bar{b}=\text{dual cost}$$

If $\bar{c}'^T\geqslant\bar{0}^T$, we have dual feasibility. And since the costs are equal, the solutions $\bar{x}$ and $\bar{p}$ are optimal for (P) and (D), respectively.

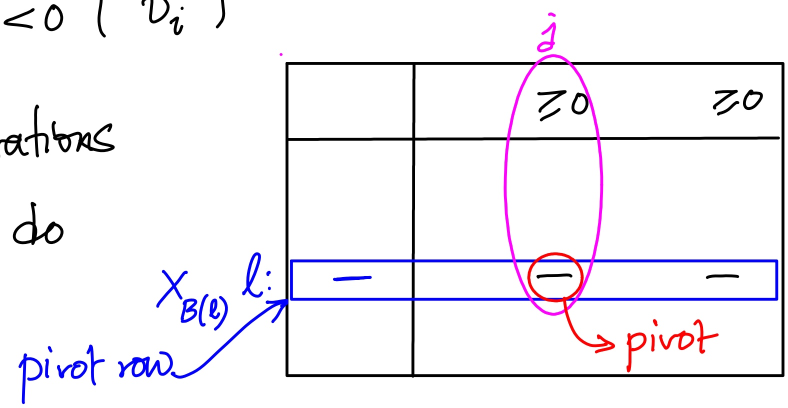
In primal simplex, we maintain primal feasibility ($B^{-1}\bar{b}\geqslant\bar{0}$), and we strive for primal optimality ($\bar{c}'^T\geqslant\bar{0}^T$). In dual simplex, we maintain dual feasibility, i.e., $\bar{c}'^T\geqslant\bar{0}^T$, and strive for dual optimality ($B^{-1}\bar{b}\geqslant\bar{0}$).

dual simplex:

$-\bar{c}_B^T\bar{x}_B$	$\bar{c}^T-\bar{c}_B^TB^{-1}A$ ← $\bar{c}'\geqslant\bar{0}$ (maintain)
could be <0 → $B^{-1}\bar{b}$	$B^{-1}A$

So, entries in Column-0 (Rows 1 to m) could be < 0 in dual simplex. If they are < 0, we "pivot them out".

Let $x_{B(\ell)} < 0$, and let the ℓ^{th} row be $(x_{B(\ell)}, v_1, \dots, v_n)$. We take this ℓth row as the **pivot row**. For all $v_i < 0$, we find $\frac{c_i'}{|v_i|} = \frac{-c_i'}{v_i}$. Let $j = \underset{v_i<0}{\arg\min} \left\{ \frac{-c_i'}{v_i} \right\}$. Then x_j enters, and $x_{B(\ell)}$ leaves. The pivoting operations then are similar to ones we do in primal simplex.



We convert column j to $\left[\frac{0}{\bar{e}_\ell}\right]$, where $\bar{e}_\ell = \ell^{th}$ unit vector.

Note that when we scale the pivot row to make the pivot entry equal to 1, the $x_{B(\ell)}$ value will necessarily become > 0, as the pivot entry $a_{\ell j}$ is necessarily < 0 to start with, and so is $x_{B(\ell)}$.

<u>Example to illustrate a dual-simplex pivot</u>

$\checkmark \frac{-6}{-2}$ $\quad \frac{-10}{-3}$ ⟶ min-ratio computations

		x_1	x_2	x_3	x_4	x_5	
	0	2	6	10	0	0	
$x_4 =$	2	-2	4	1	1	0	
$x_5 =$	-1	4	-2	-3	0	1	$R_2 \times (-\frac{1}{2})$
	-3	14	0	1	0	3	
$x_4 =$	0	6	0	-5	1	2	We now have both primal and
$x_2 =$	$\frac{1}{2}$	-2	1	$\frac{3}{2}$	0	$-\frac{1}{2}$	dual optimality!

Here is how the various pieces connect:

$$\text{primal feasibility} \xrightarrow{\text{primal simplex}} \text{primal optimality}$$
$$\quad \equiv \qquad\qquad\qquad\qquad\qquad\qquad \equiv$$
$$\text{dual optimality} \xleftarrow{\text{dual simplex}} \text{dual feasibility}$$

We take the dual path, maintaining dual feasibility and striving for dual optimality.

Consider the following LP:

$$\begin{aligned} \min\ & 5x_1 + 35x_2 + 20x_3 \\ \text{s.t.}\ & x_1 - x_2 - x_3 \le -2 \qquad x_4 \\ & -x_1 - 3x_2 \le -3 \qquad x_5 \\ & x_j \ge 0 \end{aligned}$$

slack variables: x_4, x_5

If we were to use primal simplex, we would add 2 excess and 2 artificial variables. Instead, we could start with the obvious basis using the two slack variables and do dual simplex!

		x_1	x_2	x_3	x_4	x_5
	0	5	35	20	0	0
$x_4=$	-2	1	-1	-1	1	0
$x_5=$	-3	-1	-3	0	0	1
	-15	0	20	20	0	5
$x_4=$	-5	0	-4	-1	1	1
$x_1=$	3	1	3	0	0	-1
	-40	0	0	15	5	10
$x_2=$	$5/4$	0	1	$1/4$	$-1/4$	$-1/4$
$x_1=$	$-3/4$	1	0	$-3/4$	$3/4$	$-1/4$
	-55	20	0	0	20	5
$x_2=$	1	$1/3$	1	0	0	$-1/3$
$x_3=$	1	$-4/3$	0	1	-1	$1/3$

Standard solvers such as CPLEX uses some heuristics to identify which variant (primal or dual) of simplex method to use. We often see the dual simplex being used.

Proof-type problems from Hw 7

Exercise 3.2 (Optimality conditions) Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ over a polyhedron P. Prove the following:

(a) A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}'\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

not necessarily a bfs

($\Rightarrow$) Start with optimal $\bar{x}$, show $\bar{c}^T\bar{d} \geq 0$ $\forall$ feasible directions $\bar{d}$.

($\Leftarrow$) Start with $\bar{c}^T\bar{d} \geq 0$ $\forall$ feasible directions $\bar{d}$ at $\bar{x}$, show $\bar{x}$ is optimal.

($\Rightarrow$) Let $\bar{x}$ be an optimal solution. Hence $\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y}$ $\forall\, \bar{y} \in P$.

Let $\bar{d}$ be a feasible direction at $\bar{x}$. Then there is some $\theta > 0$ such that $\underbrace{\bar{x}+\theta\bar{d}}_{\bar{y}} \in P$ (i.e., $\bar{x}+\theta\bar{d}$ is feasible).

Since $\bar{x}$ is optimal, $\cancel{\bar{c}^T\bar{x}} \leq \bar{c}^T(\cancel{\bar{x}}+\theta\bar{d}) \Rightarrow \bar{c}^T(\theta\bar{d}) \geq 0$,

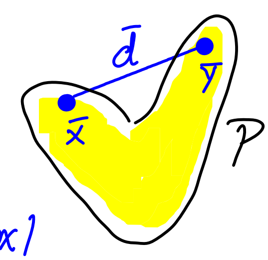
but since $\theta > 0$, we get $\bar{c}^T\bar{d} \geq 0$.

($\Leftarrow$) Let $\bar{c}^T\bar{d} \geq \bar{0}$ $\forall$ feasible directions $\bar{d}$ at $\bar{x}$. $\quad \bar{x}+\theta\bar{d} \in P$ for some $\theta > 0$.

Let $\bar{y} \in P$. We want to show $\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y}$ (i.e., $\bar{x}$ is optimal).

We write $\bar{y} = \bar{x} + (\bar{y}-\bar{x}) = \bar{x} + \theta(\bar{y}-\bar{x})$ where $\theta = 1$.

We want to argue $\bar{y}-\bar{x} (= \bar{d})$ is a feasible direction at $\bar{x}$. Indeed $\bar{y}-\bar{x}$ is a feasible direction at $\bar{x}$, as both $\bar{x}, \bar{y} \in P$ and P is convex. Writing $\bar{d} = \bar{y}-\bar{x}$, we get $\bar{c}^T\bar{d} \geq \bar{0} \Rightarrow$ $\bar{c}^T\bar{x} \leq \bar{c}^T\bar{y}$. This applies to any $\bar{y} \in P$, so $\bar{x}$ is optimal.



$\bar{d} = (\bar{y}-\bar{x})$ may not be feasible if P is not convex!

Exercise 3.3 Let $\mathbf{x}$ be an element of the standard form polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Prove that a vector $\mathbf{d} \in \Re^n$ is a feasible direction at $\mathbf{x}$ if and only if $\mathbf{A}\mathbf{d} = \mathbf{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$.

($\Rightarrow$) Let $\bar{d}$ be a feasible direction at $\bar{x}$. By definition, $\exists\, \theta > 0$ such that $\bar{x} + \theta \bar{d} \in P$. $\bar{x} \in P$ to start with, so $A\bar{x} = \bar{b}$. Also, we have $A(\bar{x} + \theta \bar{d}) = \bar{b}$. Hence $\theta A\bar{d} = \bar{0}$, which along with $\theta > 0$ gives $A\bar{d} = \bar{0}$. Further, $\bar{x} + \theta \bar{d} \geq \bar{0}$, which gives $\theta d_i \geq 0$ when $x_i = 0$, and since $\theta > 0$, we get $d_i \geq 0$ $\forall i$ s.t. $x_i = 0$.

($\Leftarrow$) Let $\bar{d}$ be such that $A\bar{d} = \bar{0}$ and $d_i \geq 0$ $\forall i$ with $x_i = 0$.

We have $A(\bar{x} + \theta \bar{d}) = A\bar{x} = \bar{b}$ for any θ, as $\bar{x} \in P$.

Further, when $x_i = 0$, $x_i + \theta d_i \geq 0$ for all $\theta \geq 0$. When $x_i > 0$, $x_i + \theta d_i \geq 0$ for small enough $\theta > 0$ (think min-ratio test).

Hence for some $\theta > 0$, $\bar{x} + \theta \bar{d} \in P$, i.e., $\bar{d}$ is a feasible direction at $\bar{x}$.

Since $x_i > 0$, assume $d_i < 0$ (else the result is trivial). We are scaling d_i by $\theta > 0$, though, and hence $x_i + \theta d_i \geq 0$ for θ small enough. Now, we consider all $x_i > 0$, and take the smallest $\theta > 0$ for all i.

This result gives a characterization of all feasible directions $\bar{d}$ at $\bar{x}$: every such direction satisfies $A\bar{d} = \bar{0}$ and $d_i \geq 0$ when $x_i = 0$.