MATH464 – Lecture 27 (04/18/2023)

Today: * Bland's rule in Matlab
* HW and other problems from BT-ILO

Bland's rule

Simply put, this is the "minimum index rule"— the non-basic x_j with $c_j' < 0$ and smallest j enters, and in case of a tie, the basic variable x_ℓ with the smallest ℓ (that ties) leaves. Since we order the variables $x_1, \dots, x_n$ by default, choosing the entering variable is easy—just pick the leftmost one. But choosing the leaving variable, while easy to do by hand, will require a bit more work.

We will maintain and update the basis $\mathcal{B}$ (stored as Bind in Matlab). See the session from today's lecture for details:

https://www.math.wsu.edu/faculty/bkrishna/FilesMath464/S23/Software/Lec27_04182023_Session.txt

Now let's consider a problem not assigned in the homework.

Exercise 3.19 While solving a standard form problem, we arrive at the following tableau, with x_3, x_4, and x_5 being the basic variables:

			x_2			
	-10	δ	$-2 \, {=}0$	0	0	$0 \to \frac{2}{3}$
	4	-1	η	1	0	0
x_4	1	α	-4	0	1	0
	$\beta{=}0$	γ	3	0	0	1

$R_0 + \left(\frac{2}{3}\right) R_3$

$\delta + \left(\frac{2}{3}\right)\gamma$

The entries α, β, γ, δ, η in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.

(a) The current solution is optimal and there are multiple optimal solutions.

(b) The optimal cost is $-\infty$.

(c) The current solution is feasible but not optimal.

(a) Notice x_2 could enter (as $c_2' = -2 < 0$). Thus the current bfs is optimal only if x_2 could enter but not change the cost. In other words, we need a degenerate bfs here. If $\beta = 0$ and $\gamma > 0$, we could get $\theta^* = 0$ (min-ratio). Hence x_2 could enter without changing the cost. The EROs give $c_1' = \delta + \left(\frac{2}{3}\right)\gamma$. If $\delta + \left(\frac{2}{3}\right)\gamma \geq 0$, the resulting tableau is optimal, and hence so is the current solution, thus giving multiple optimal solutions. $\boxed{\beta = 0, \gamma > 0, \delta + \frac{2}{3}\gamma \geq 0.}$

(b) x_2 cannot improve the cost without bound, as we need $\beta \geq 0$ for feasibility. We get unbounded LP with $\delta < 0$, $\alpha \leq 0$, $\gamma \leq 0$ ($\beta \geq 0$ is needed for feasibility).

(c) We need $\beta > 0$, as then we get $\theta^* = \min\left(\frac{\beta}{3}, \frac{4}{\eta}\right)$ if $\eta > 0$. $\theta^* > 0$ here, and hence x_2 could enter to improve the solution. We also need either $\delta \geq 0$, or if $\delta < 0$, then $\alpha > 0$ or $\gamma > 0$. Another option is $\delta < 0$, and $\alpha < 0$, $\gamma < 0$, when the LP will be unbounded.

Hw10 Problems

Exercise 4.3 The purpose of this exercise is to show that solving linear programming problems is no harder than solving systems of linear inequalities.

Suppose that we are given a subroutine which, given a system of linear inequality constraints, either produces a solution or decides that no solution exists. Construct a simple algorithm that uses a single call to this subroutine and which finds an optimal solution to any linear programming problem that has an optimal solution.

Subroutine : feasibility of a system of linear inequalities.

Use LP duality to come up with a system of linear inequalities that you could input to the subroutine only once.

(P) $\min \bar{c}^T\bar{x}$ s.t. $A\bar{x} = \bar{b}$, $\bar{x} \geq \bar{0}$

$\max \bar{p}^T\bar{b}$ s.t. $\bar{p}^T A \leq \bar{c}$ (D)

Hint: weak/strong duality.

$\hookrightarrow$ $\bar{c}^T\bar{x} \geq \bar{p}^T\bar{b}$ for feasible $\bar{x}, \bar{p}$ for (P) and (D).

min $\bar{c}^T\bar{x}$

optimality $\times$

max $\bar{p}^T\bar{b}$

If $\bar{c}^T\bar{x} = \bar{p}^T\bar{b}$, then $\bar{x}, \bar{p}$ are optimal for (P) and (D), respectively.

Consider the system

$$\boxed{\begin{array}{c} \left.\begin{array}{c} A\bar{x} = \bar{b} \\ \bar{x} \geq \bar{0} \end{array}\right\} \text{(P) feasibility} \\ \bar{p}^T A \leq \bar{c} \longrightarrow \text{(D) feasibility} \\ \bar{c}^T\bar{x} = \bar{p}^T\bar{b} \longrightarrow \text{optimality} \end{array}}$$

Call subroutine once with this system as input.

Bonus: Think about how to handle the situation if the subroutine says the system is infeasible!

Farkas' lemma:

Exercise 4.26 Let $\mathbf{A}$ be a given matrix. Show that exactly one of the following alternatives must hold.

(a) There exists some $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$.

(b) There exists some $\mathbf{p}$ such that $\mathbf{p}'\mathbf{A} > \mathbf{0}'$.

we want an equivalent system that is $\geq$, not $>$.

The crux is to come up with an appropiate pair of primal-dual LPs, and follow arguments presented in class.

$$(P) \quad \begin{aligned} \max\ & \bar{1}^T \bar{x} \\ & A\bar{x} = \bar{0} \\ & \bar{x} \geq \bar{0} \end{aligned} \qquad\qquad \begin{aligned} \min\ & \bar{p}^T \bar{0} \\ & \bar{p}^T A \geq \bar{1} \end{aligned} \quad (D)$$

Suppose (b) holds. Then (D) is feasible. (D) is not unbounded, as $\bar{p}^T \bar{0} = 0$ for any $\bar{p}$. So (D) has optimal cost $= 0$. Hence for (P), we get that the only optimal solution is $\bar{x} = \bar{0}$. (as $\bar{1}^T \bar{x} = \sum_i x_i = 0$ optimal $\Rightarrow$ $\bar{x} = \bar{0}$ is the only optimal solution).

So (a) cannot hold.