MATH 464 – Lecture 2 (01/12/2023)

Today: * general form of LP
* standard form of LP

Homework 1 is posted:
* present arguments that work in general — illustration on small examples is not sufficient

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Definitions of Linear Programs (LPs) Forms

General Form of LP:

$$\begin{aligned} \min\ & z = \bar{c}^T \bar{x} \\ \text{s.t.}\ & \bar{a}_i^T \bar{x} \geq b_i, \quad i \in M_1 \\ & \bar{a}_i^T \bar{x} \leq b_i, \quad i \in M_2 \\ & \bar{a}_i^T \bar{x} = b_i, \quad i \in M_3 \end{aligned}$$

"is element of"

M_1, M_2, M_3: subsets of indices from $\{1, 2, \ldots, m\}$

Sign restrictions $\begin{cases} x_j \geq 0, & j \in N_1 \\ x_j \leq 0, & j \in N_2 \\ x_j \text{ urs}, & j \in N_3 \end{cases}$

N_1, N_2: subsets of indices from $\{1, 2, \ldots, n\}$

Vectors are columns by default (similar to how they are set up in Matlab).

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the vector of decision variables (d.v.'s). Each d.v. x_j is either ≥ 0, ≤ 0 or unrestricted in sign (urs).

Notice $M_1 \cup M_2 \cup M_3 = \{1, 2, \ldots, m\}$, but $N_1 \cup N_2$ need not be $\{1, 2, \ldots, n\}$. We could add x_j urs, $j \in N_3$ as a last sign restriction, and then get $N_1 \cup N_2 \cup N_3 = \{1, 2, \ldots, n\}$.

$\bar{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is the objective function coefficients vector.

$\bar{c}^T\bar{x}$ is the objective function.

$\bar{a}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$ is the constraint coefficient vector for the i^{th} constraint.

Stacking the $\bar{a}_i$ vectors as rows gives the $m \times n$ matrix A.

b_i is the right-hand side (rhs) coefficient of i^{th} constraint.

Illustration on Dude's LP:

$$\left.\begin{array}{lrl} \text{max} & z = 2x_1 + 3x_2 & \text{(total utility)} \\ \text{s.t.} & x_1 + x_2 \leq 5 & \text{(max time)} \\ & 8x_1 + 16x_2 \leq 48 & \text{(max money)} \\ & x_1 \geqslant 0,\ x_2 \geqslant 0 & \text{(nonnegativity)} \end{array}\right\} \text{Linear Program (LP)}$$

max → maximize

s.t. → subject to

$$\max\ \bar{c}^T\bar{x} \ \equiv\ \min\ -\bar{c}^T\bar{x}$$

$\equiv$ "equivalent to"

We could minimize $-\bar{c}^T\bar{x}$ when we have to maximize $\bar{c}^T\bar{x}$.
We could equivalently define the standard form for a maximization LP.

Hence $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\bar{c} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $\bar{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 48 \end{bmatrix}$

$M_1 = \phi$ (empty set), $M_2 = \{1,2\}$, $M_3 = \phi$, $N_1 = \{1,2\}$, $N_2 = \phi$, $N_3 = \phi$.

Def If $\bar{x}$ satisfies all constraints (including sign restrictions), it is called a **feasible solution** or feasible vector.

If $\bar{x}^*$ is a feasible solution such that $\bar{c}^T\bar{x}^* \le \bar{c}^T\bar{x}$ $\forall$ feasible $\bar{x}$, then $\bar{x}^*$ is an **optimal solution**.

"for all"

With some abuse of notation, we write the general form LP as

$$\begin{array}{ll} \min & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x} \begin{pmatrix} \ge \\ \le \\ = \end{pmatrix} \bar{b} \\ & \text{sign restrictions } \bar{x} \end{array}$$

We are moving toward the use of results from Linear Algebra on solving $A\bar{x}=\bar{b}$.

With this notation, for the Dude LP, we get

$$A = \begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 5 \\ 48 \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{ and } \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Here is another example:

$$\begin{array}{lrl} \min & 2x_1 + 3x_2 - x_3 & \\ \text{s.t.} & x_1 + x_2 & \ge 4 \\ & 3x_1 - x_2 + 5x_3 & \le 1 \\ & 4x_2 + 3x_3 & = 6 \\ & x_1 \ge 0, \ x_2 \le 0, \ x_3 \text{ urs} & \end{array}$$

if not specified, a variable is assumed to be urs.

Standard Form of LP

$$\boxed{\begin{array}{ll} \min & \bar{c}^T \bar{x} \\ \text{s.t.} & A\bar{x} = \bar{b} \\ & \bar{x} \geq \bar{0} \end{array}}$$

all variables are non-negative, all constraints are equations.

Recall that we have learned how to solve the general system $A\bar{x}=\bar{b}$. We will use that knowledge to solve any LP, but describe the method for this standard form. Of course, we can convert any general LP to its equivalent standard form LP.

Conversion to Standard Form

1. If $x_j \leq 0$, then replace x_j with $-x_j'$, and $x_j' \geq 0$.

   replace → every occurrence of x_j with $-x_j'$

   $-3 \longrightarrow -(\underset{\geq 0}{3})$

   e.g., $x_2 \leq 0 \Rightarrow -x_2 \geq 0$.

   so, $x_2 \longrightarrow -x_2'$ and add $x_2' \geq 0$ (we will replace every occurrence of x_2 by $(-x_2')$).

   "replace"

2. If x_j is urs, replace x_j by $x_j^+ - x_j^-$, add $x_j^+, x_j^- \geq 0$.

   $-3 = 0-3 \longrightarrow x_j^- = 3$ here

   but $-3 = 5-8$, for instance $\longrightarrow$ but in an optimal solution, we will have only one of $x_j^+, x_j^- > 0$.

   e.g., x_3 urs $\longrightarrow$ $x_3 \rightarrow x_3^+ - x_3^-$, $x_3^+, x_3^- \geq 0$

x_3^+, x_3^- capture the positive and negative part of x_3. Depending on the value of x_3, only one of x_3^+ and x_3^- will be positive. For instance, if $x_3 = 2$, then $x_3^+ = 2$, $x_3^- = 0$; and if $x_3 = -5$, then $x_3^+ = 0$, $x_3^- = 5$.

The result that both x_i^+ and x_i^- cannot be >0 follows from elementary linear algebra properties. Recall the notion of basic variables (and free or non-basic variables) in the solution of $A\bar{x}=\bar{b}$. The variables that are >0 correspond to basic variables, which in turn correspond to pivot columns of A. But columns of x_i^+ and x_i^- are just (-1) multiples of each other — and hence are linearly dependent. So, both cannot be pivot columns at the same time.

3. If **constraint i is $\geq$**, subtract an **excess variable** e_i from the left-hand side (lhs), and add $e_i \geq 0$.

e.g., $x_1 + x_2 \geq 4 \longrightarrow x_1 + x_2 - e_1 = 4, \; e_1 \geq 0$

e_1 captures the amount by which x_1+x_2 exceeds 4. Hence we must insist $e_1 \geq 0$. If $e_1 = -2$, for instance, $x_1+x_2=2$, which violates the original constraint.

4. If **constraint i is $\leq$**, add **slack variable** s_i to the left-hand side (lhs), and add $s_i \geq 0$.

e.g., $3x_1 - x_2 + 5x_3 \leq 1$ is replaced by $3x_1 - x_2 + 5x_3 + s_2 = 1$, $s_2 \geq 0$.

We apply these transformations to convert the second LP example to standard form. ↳ after Dude's LP.

$$
\begin{array}{ll}
\min & 2x_1 + 3x_2 - \overset{(x_3^+ - x_3^-)}{x_3} \\
\text{s.t.} & x_1 + x_2 - e_1 \geq 4 \\
 & 3x_1 - x_2 + 5x_3 + s_2 \leq 1 \\
 & 4x_2 + 3x_3 = 6 \\
 & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs}
\end{array}
$$

x_2', $x_2' \geq 0$

$$
\begin{array}{ll}
\min & 2x_1 - 3x_2' - (x_3^+ - x_3^-) \\
\text{s.t.} & x_1 - x_2' - e_1 = 4 \\
 & 3x_1 + x_2' + 5(x_3^+ - x_3^-) + s_2 = 1 \\
 & -4x_2' + 3(x_3^+ - x_3^-) = 6 \\
 & \text{all vars} \geq 0
\end{array}
$$

(LP in standard form)

Could use x_4, x_5, x_6, etc. instead of x_2', x_3^+, x_3^-, for instance.

Note that we do not have to do anything extra for variables that are ≥ 0 already, and for constraints that are equations.