MATH464 – Lecture 6 (01/26/2023)

Today: * graphical solution of LPs in 2D
* Cases of LP

How to solve LPs in 2D Graphically

Consider the following LP in 2D:

$$
\begin{aligned}
\min\ & 2x_1+x_2\\
\text{s.t. } & x_1+x_2 \geq 2\\
& 3x_1+x_2 \geq 4\\
& 3x_1+2x_2 \leq 10\\
& x_1, x_2 \geq 0
\end{aligned}
$$

We demonstrate how we can "plot" this LP, and solve it in that process. Later on, we will extend these techniques to higher dimensions using linear algebra.

The set of all points (x_1,x_2) which satisfy all constraints (including nonnegativity) is called the **feasible region** of the LP.

We first plot the feasible region. To do so, we plot each inequality.

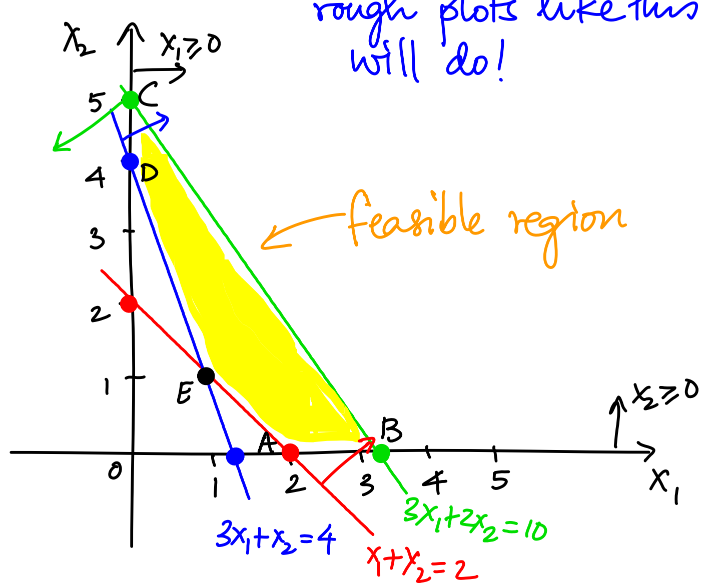
How to plot $x_1+x_2 \geq 2$? We first plot $x_1+x_2=2$. To plot any line, you need two points. The easiest choices are to pick $x_1=0$ and then $x_2=0$ to get the two points, which are $(0,2)$ and $(2,0)$. $x_1+x_2=2$ divides the plane into two half-planes. We need to pick the correct one that is represented by $x_1+x_2 \geq 2$.

To do so, pick any point, say, $(0,0)$ and test it on $x_1+x_2 \geq 2$.

$$0+0 \not\geq 2$$

So $(0,0)$ is on the wrong side. Hence we pick the other side (not containing $(0,0)$), and indicate the choice by drawing the arrow $\rightarrow$ to that side from the line of $x_1+x_2=2$.

rough plots like this will do!

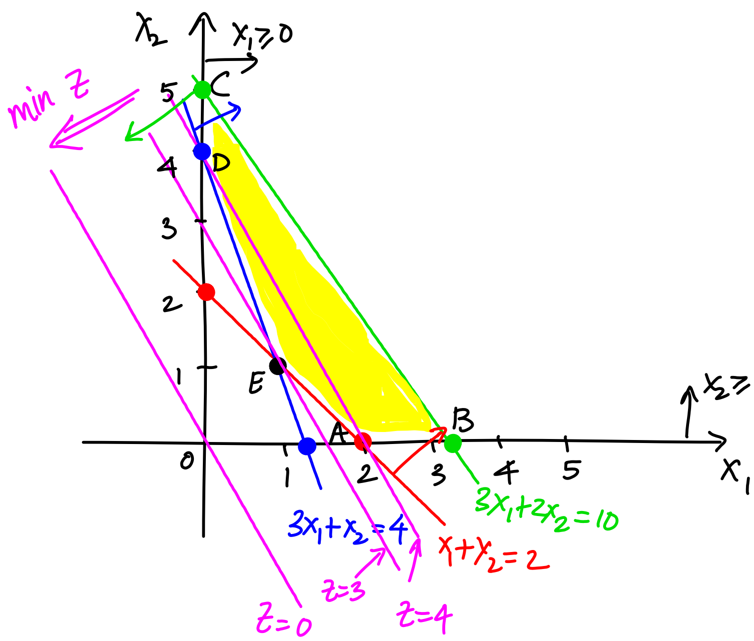


Once we have plotted all the inequalities, including $x_1 \geq 0$ and $x_2 \geq 0$, the region that is the intersection of all half planes is shaded. This is indeed the feasible region, the polygon ABCDE here.

While it appears E has coordinates (1,1) from the plot, it's better to actually solve for its coordinates, i.e., solve the linear system $\left\{ \begin{array}{l} x_1+x_2=2 \\ 3x_1+x_2=4 \end{array} \right\}$, which indeed gives us (1,1) as the coordinates.

What about the objective function min $2x_1+x_2$?

Denoting $2x_1+x_2=z$, we plot this line for at least one value of z (may be two), to decide which way to slide it so as to decrease z (recall that we are minimizing z).



Plot the obj. fn for two values of z.

e.g., $z=4$: $2x_1+x_2=4$

We slide the z-line down (min), until we hit E(1,1), where $z=2(1)+(1)=3$. We cannot slide it any further, as we will go out of the feasible region.

So E(1,1), or $x_1=1, x_2=1$ is the optimal solution, giving the optimal objective function value $z^*=2(1)+(1)=3$.

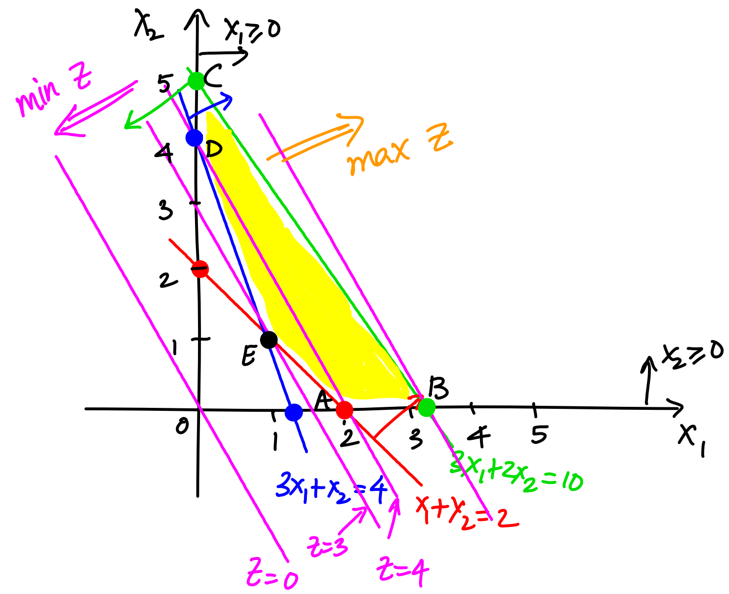
max
~~min~~ $z = 2x_1 + x_2$

s.t. $x_1 + x_2 \geq 2$

$3x_1 + x_2 \geq 4$

$3x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$



Here, $E(1,1)$ is the unique optimal solution. To make sure, we check z value at $A(2,0)$ and $D(0,4)$: $z(A) = 2(2)+1(0) = 4$ and $z(D) = 2(0)+1(4) = 4$, both are $> z(E) = 3$.

In the case of a system of linear equations $A\bar{x} = \bar{b}$, we have three possibilities— the system has a unique solution, it has infinitely many solutions, or it is inconsistent. We get corresponding cases for LP, but get one more case in addition.

This LP is an example of **Case I**, or **Type I**, where the LP has a **unique optimal solution**.

Consider a small variation now:

For $\left\{\begin{array}{l} \max\ z = 2x_1 + x_2 \\ \text{s.t. same constraints} \end{array}\right\}$ $B(10/3, 0)$ is the unique optimal solution.

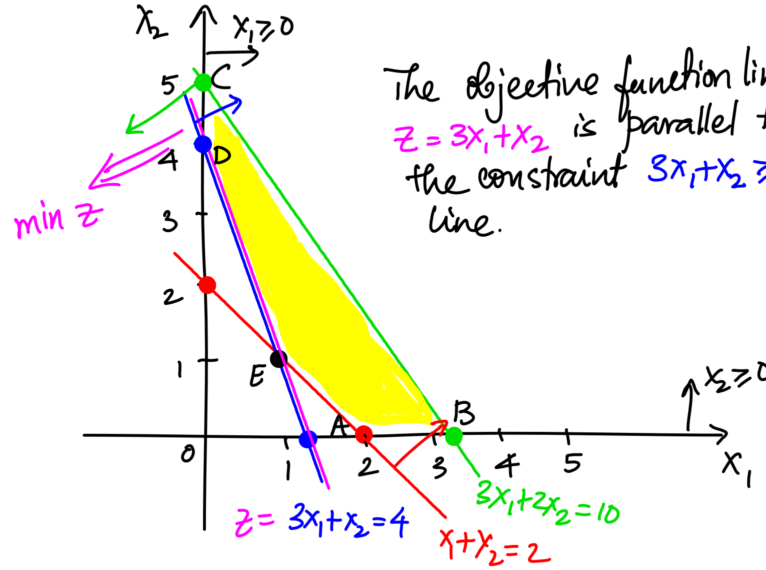
This max-LP also is of Case I.

$z(B) = 2(10/3) + 1(0) = \frac{20}{3}$.

$z(C) = 2(0) + 1(5) = 5 < \frac{20}{3}$.

Consider a slightly different LP:

$$\min\ z = 3x_1 + x_2$$
s.t.
$$x_1 + x_2 \geq 2$$
$$3x_1 + x_2 \geq 4$$
$$3x_1 + 2x_2 \leq 10$$
$$x_1, x_2 \geq 0$$



The objective function line $z = 3x_1 + x_2$ is parallel to the constraint $3x_1 + x_2 \geq 4$ line.

Here, the z-line hits the entire line segment $\overline{DE}$. Every point in the line segment is an optimal solution. So the LP has **alternative optimal solutions**, and belongs to **Case II**.

Recall: parametric vector form of solutions to $A\bar{x} = \bar{b}$.
For instance, if there were two free variables, we could write all solutions in the form $\bar{x} = \bar{x}_0 + p\bar{r} + q\bar{s}$, where $p, q \in \mathbb{R}$, and $A\bar{x}_0 = \bar{b}$, i.e., $\bar{x}_0$ is a particular solution, and $A\bar{r} = \bar{0}$, $A\bar{s} = \bar{0}$, i.e., both $\bar{r}$ and $\bar{s}$ belong to $\mathrm{Nul}(A)$.

Notice that $E(1,1)$ and $D(0,4)$ both give $z^* = 3x_1 + x_2 = 4$.

We can describe all optimal solutions to this LP as

$$\bar{x} = \lambda \underbrace{\begin{bmatrix} 0 \\ 4 \end{bmatrix}}_{D} + (1-\lambda) \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{E}, \quad \lambda \in [0,1] \quad \text{i.e.,} \quad \bar{x} = \begin{bmatrix} 1-\lambda \\ 3\lambda+1 \end{bmatrix}, \quad \lambda \in [0,1].$$

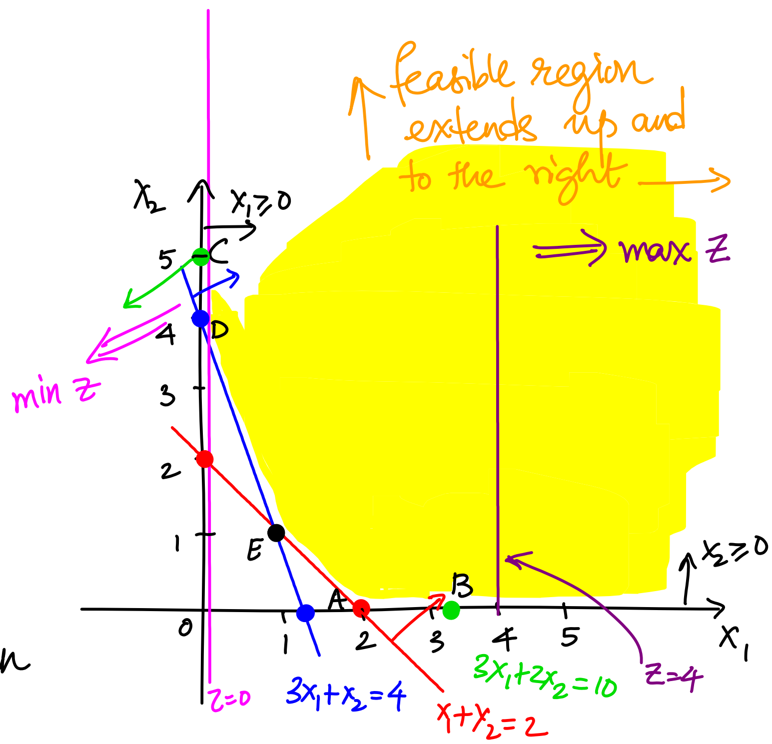
Indeed, $\bar{c}^T \bar{x} = [3 \ \ 1] \begin{bmatrix} 1-\lambda \\ 3\lambda+1 \end{bmatrix} = 3 - 3\lambda + 3\lambda + 1 = 4.$

$\bar{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the objective function coefficient vector

In higher dimensions, we could get more that two "vertices" (such as D & E here), and the entire "face" between them as the set of optimal solutions.

We consider another LP of Case II below.

$$\min \quad x_1$$
$$\text{s.t.} \quad x_1+x_2 \geq 2$$
$$3x_1+x_2 \geq 4$$
$$x_1, x_2 \geq 0$$



Here, any point on the vertical axis at or above D(0,4) is an optimal solution.

Here, the optimal solution set is the ray going vertically up from D(0,4) (or the half-open line from D(0,4) up). While we again have infinitely many optimal solutions here, notice that the optimal solution set is **unbounded**, unlike $\overline{DE}$ in the previous case.

We can describe all optimal solutions here as

$$\bar{x} = \underbrace{\begin{bmatrix}0\\4\end{bmatrix}}_{D} + \lambda \begin{bmatrix}0\\1\end{bmatrix}, \lambda \in \mathbb{R}, \lambda \geq 0.$$

→ the unit vector pointing up along the vertical axis

We now consider a slight modification of the above LP.

$$\max \quad x_1$$
$$\text{s.t.} \quad x_1+x_2 \geq 2$$
$$3x_1+x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

$z=x_1$ line could be pushed to the right without limit, i.e., there is no finite optimal z-value. We say the LP is **unbounded**, and it belongs to **Case III**.

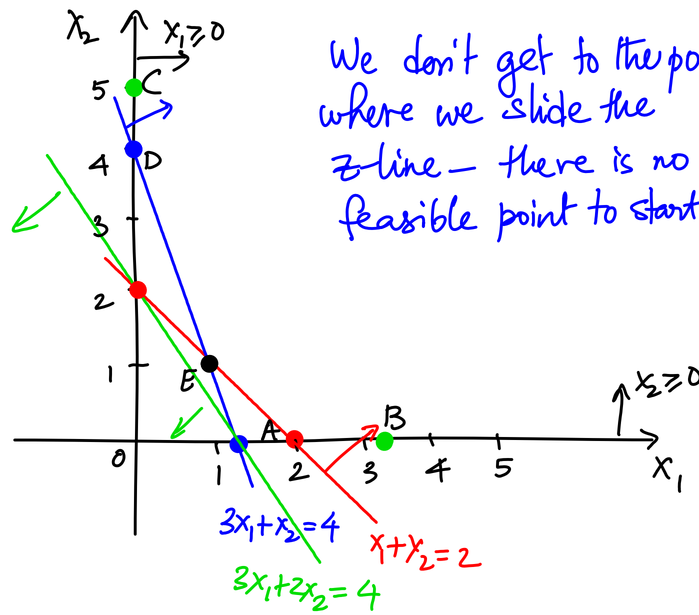
This is the extra case which does not have a corresponding case in $A\bar{x}=\bar{b}$ (system of linear equations).

Notice the difference between unbounded feasible region and unbounded LP. If the feasible region is bounded, the LP cannot be unbounded as the z-line cannot be moved in the improving direction without limits. On the other hand, the feasible region could be unbounded but the LP could still have an optimal solution, as seen above (variation of Case II).

4. Case IV LP (infeasible LPs)

Consider a slightly different LP:

$$\begin{aligned} \min\ z = & \ 3x_1 + x_2 \\ \text{s.t.}\quad & x_1 + x_2 \geq 2 \\ & 3x_1 + x_2 \geq 4 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$



We don't get to the point where we slide the z-line – there is no feasible point to start!

There are no feasible solutions, i.e., there are no points satisfying all constraints. So the LP is **infeasible**. So its feasible region is empty.

In practice, if an LP formulation comes out to be infeasible, it could indicate that, for instance, we cannot meet all demand using the resources available. Or that the cost for a particular necessary project will not fit within the budget.