MATH 464 – Lecture 7 (01/31/2023)

Today: * Results on polyhedra

Background in Linear Algebra and Geometry

We will present definitions and results related to polyhedra and convexity. The goal is to generalize the graphical solution of LP in 2D to high dimensions.

Subspaces: A set $S \subseteq \mathbb{R}^n$ is a subspace if $\forall\, \bar{x}, \bar{y} \in S$,

$a\bar{x} + b\bar{y} \in S$ for all $a, b \in \mathbb{R}$.

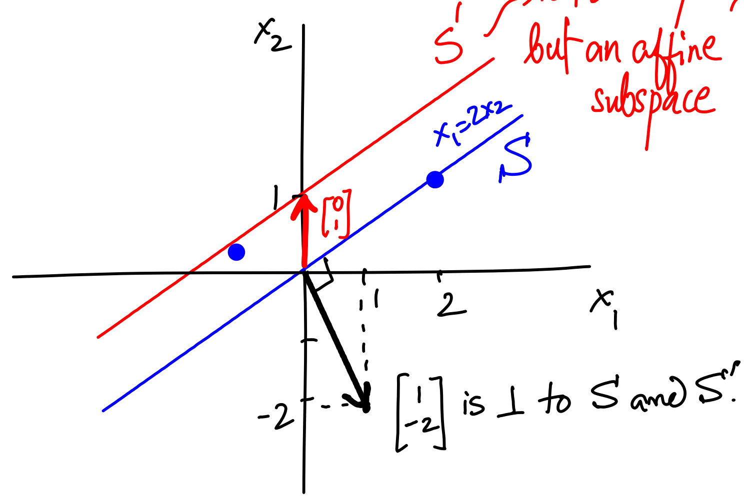
In particular, $\bar{0} \in S$ (for $a=b=0$). → we usually mention the inclusion of $\bar{0}$ separately (to define a subspace).

$\uparrow$ zero vector

e.g. in 2D, any line passing through the origin is a subspace of $\mathbb{R}^2$

$x_1 = 2x_2$ or $x_1 - 2x_2 = 0$, i.e.,

$$\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$



All points on S can be described as $\lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $\lambda \in \mathbb{R}$.

So, $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for S, and $\dim S = 1$ (dimension of S).

If S is strictly smaller than $\mathbb{R}^n$, we say that S is a proper subspace of $\mathbb{R}^n$.

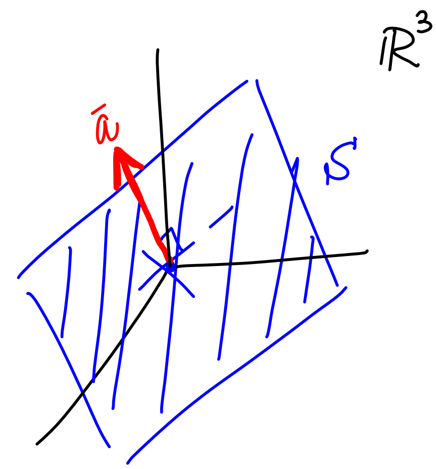
In $\mathbb{R}^3$, planes passing through the origin are proper subspaces.

Affine subspace: Given a subspace S of $\mathbb{R}^n$, the set $S_0 = \bar{x}_0 + S = \{\bar{x}_0 + \bar{x} \mid \bar{x} \in S\}$ is an affine subspace for $\bar{x}_0 \notin S$.

Each proper subspace and affine subspace in $\mathbb{R}^2$ has a vector orthogonal to it. In higher dimensions, if S is an m-dimensional subspace of $\mathbb{R}^n$ with $m < n$, there will be $n-m$ linearly independent (LI) vectors orthogonal to S.

In the 2D example above, $\bar{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is orthogonal to both $S = \{\bar{x} \mid \bar{x} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \lambda \in \mathbb{R}\}$ and $S' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + S$.

Illustration in $\mathbb{R}^3$ for the case when $\dim S = 2$.



Back to feasible region of LP...

We now introduce polyhedra, which are generalizations of "nice," i.e., convex polygons in 2D to higher dimensions. The feasible region of an LP is a polyhedron.

≥ 0 are included

Def A **polyhedron** is $P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \geq \bar{b}\}$, where $A_{m \times n}$, $\bar{b}_{m \times 1}$ have real entries.

(m, n are finite here)

$\bar{a}_i^T \bar{x} \geq b_i$ for $i = 1, \ldots, m$

m linear inequalities

In words, P is a set of points satisfying a finite set of linear inequalities. If there are non-negativity constraints involved, they are included as part of the main set of constraints $A\bar{x} \geq \bar{b}$.

If there are equations of the form $\bar{a}_i^T\bar{x} = b_i$, we could split each of them into a pair of inequalilities of the form $\bar{a}_i^T\bar{x} \geq b_i$ and $\bar{a}_i^T\bar{x} \leq b_i$, and rewrite the latter inequalities as $-\bar{a}_i^T\bar{x} \geq -b_i$.

Hyperplane: $\{\bar{x} \in \mathbb{R}^n \mid \bar{a}^T\bar{x} = b\}$, $\bar{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$

e.g., $ax+by+cz=d$ is a plane in 3D

Half-space: $\{\bar{x} \in \mathbb{R}^n \mid \bar{a}^T\bar{x} \geq b\}$.

Result The feasible region of an LP is a polyhedron.

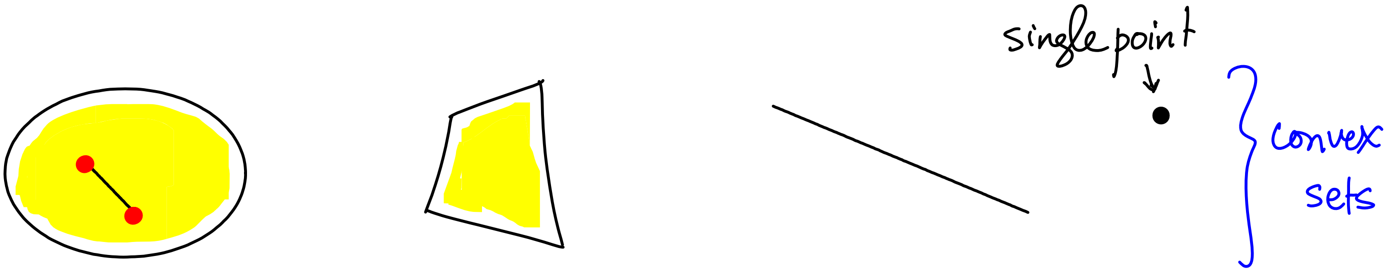
Result A polyhedron is the intersection of half-planes.

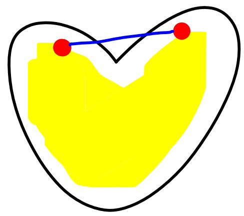
Recall how we drew the feasible region in 2D. After selecting the correct side of each equation (indicated by the arrows $\rightarrow$ or $\leftarrow$), we pick the region common to all the selected half-spaces.

We previosly defined convex sets:

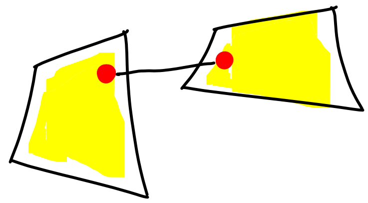
Def A set $S \subseteq \mathbb{R}^n$ is a **convex set** if for all $\bar{x}, \bar{y} \in S$ and $\lambda \in [0,1]$, $\underbrace{\lambda\bar{x} + (1-\lambda)\bar{y}}_{\text{convex combination of } \bar{x} \text{ and } \bar{y}} \in S$.

In words, the line segment connecting any two points in the set lies entirely within the set.

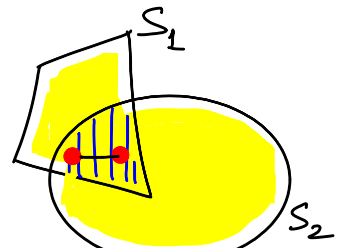




not convex!



union of convex sets need not be convex



But intersections of convex sets are convex

BT-ILO Theorem 2.1 A polyhedron is a convex set.

Proof We present the proof by showing two results.

(a) Intersection of convex sets is convex.

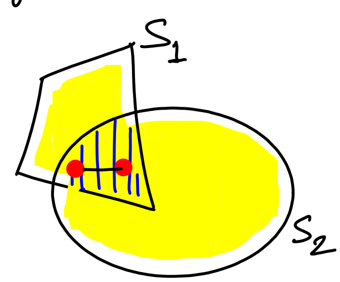
Let $S_i, i \in I$ be convex, and $S = \bigcap_{i \in I} S_i$, the intersection of S_i.

Let $\bar{x}, \bar{y} \in S \Rightarrow \bar{x}, \bar{y} \in S_i \;\forall i \in I$.

Since S_i is convex, $\lambda \bar{x} + (1-\lambda)\bar{y} \in S_i$ for $\lambda \in [0,1]$ $\forall i \in I$.

$$\Rightarrow \lambda \bar{x} + (1-\lambda)\bar{y} \in S \quad (\text{as } S = \bigcap_{i \in I} S_i)$$

$$\Rightarrow S \text{ is convex.}$$



I need not be finite here

(b) Half space $H = \{\bar{x} \in \mathbb{R}^n \mid \bar{a}^T \bar{x} \geq b\}$ is convex.

Let $\bar{x}, \bar{y} \in H \Rightarrow \bar{a}^T \bar{x} \geq b$ and $\bar{a}^T \bar{y} \geq b$

$$\Rightarrow \bar{a}^T(\lambda \bar{x} + (1-\lambda)\bar{y}) = \lambda \bar{a}^T \bar{x} + (1-\lambda)\bar{a}^T \bar{y} \quad \text{for } \lambda \in [0,1]$$

$$\geq \lambda b + (1-\lambda) b = b$$

$$\Rightarrow \lambda \bar{x} + (1-\lambda)\bar{y} \in H, \text{ i.e., } H \text{ is convex.}$$

Polyhedron P is the intersection of m half-spaces $H_i = \{\bar{x} \in \mathbb{R}^n \mid \bar{a}_i^T \bar{x} \geqslant b_i\}$.
Then by (a) and (b) above, we get that P is convex □

Alternatively, we could present a more direct proof. Let $P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \geqslant \bar{b}\}$, and let $\bar{x}, \bar{y} \in P$. Then $A\bar{x} \geqslant \bar{b}$ and $A\bar{y} \geqslant \bar{b}$. Hence

$$A(\lambda\bar{x} + (1-\lambda)\bar{y}) \geqslant \lambda\bar{b} + (1-\lambda)\bar{b} = \bar{b} \implies \lambda\bar{x} + (1-\lambda)\bar{y} \in P \text{ as well.}$$

Some more definitions

Def A set $S \subset \mathbb{R}^n$ is **bounded** if $\max\limits_{i=1,\dots,n} \{|x_i| \mid \bar{x} \in S\} \leq K$ for some finite non-negative number K.

A bounded polyhedron is called a **polytope**.

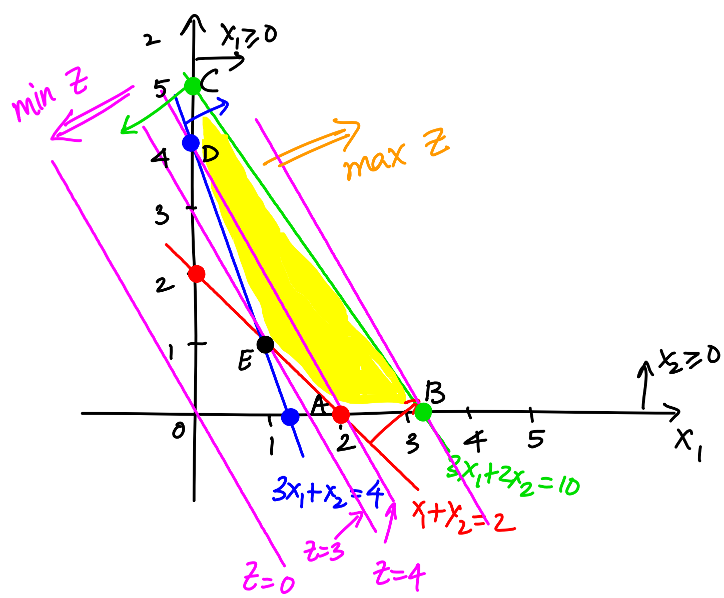
$\cancel{\min}$ max $z = 2x_1 + x_2$

s.t. $x_1 + x_2 \geqslant 2$

$3x_1 + x_2 \geqslant 4$

$3x_1 + 2x_2 \leq 10$

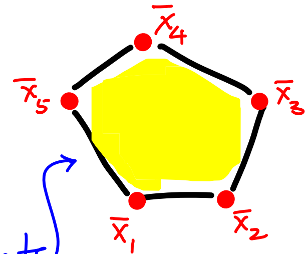
$x_1, x_2 \geqslant 0$



The feasible region ABCDE is bounded, e.g., $K=6$ will work.
But if we remove $3x_1 + 2x_2 \leq 10$, the feasible region is no longer bounded.

Recall A convex combination of $\bar{x}_1, \ldots, \bar{x}_m \in \mathbb{R}^n$ is

$$\bar{x} = \sum_{i=1}^{m} \lambda_i \bar{x}_i, \quad \lambda_i \geq 0 \ \forall i, \quad \sum_{i=1}^{m} \lambda_i = 1.$$



strap an elastic band around all points

Def The **convex hull** of $\bar{x}_1, \ldots, \bar{x}_m = \{\bar{x} \in \mathbb{R}^n \mid \bar{x} = \sum_{i=1}^{m} \lambda_i \bar{x}_i, \ \lambda_i \geq 0, \ \sum_i \lambda_i = 1\}$.

denoted $\operatorname{conv}(\bar{x}_1, \ldots, \bar{x}_m)$.

e.g., The convex hull of A, B, C, D, E is the feasible region of the LP.

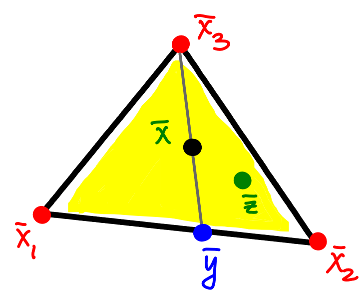
BT-ILO Theorem 2.1(d) The convex hull of $\bar{x}_1, \ldots, \bar{x}_m \in \mathbb{R}^n$ is a convex set.

Read proof in book! Intuitive idea is presented here.

Could use an "inductive" argument. Consider $\operatorname{conv}(\{\bar{x}_1, \bar{x}_2\})$.

$\bar{y} = \lambda \bar{x}_1 + (1-\lambda)\bar{x}_2$ for some $\lambda \in [0,1]$. Any point in $\operatorname{conv}(\{\bar{x}_1, \bar{x}_2, \bar{x}_3\})$ can be written as

$$\bar{x} = \mu \bar{y} + (1-\mu)\bar{x}_3 \text{ for } \mu \in [0,1].$$



Now consider another point $\bar{z} \in \operatorname{conv}(\{\bar{x}_1, \bar{x}_2, \bar{x}_3\})$. We can show that $\eta \bar{x} + (1-\eta)\bar{z} \in \operatorname{conv}(\{\bar{x}_1, \bar{x}_2, \bar{x}_3\})$.

Recall how E(1,1), the optimal solution in the previous LP, was a vertex or corner point of the feasible region ABCDE. Indeed, we can generalize this observation — we can look of "vertices" of feasible regions for candidate optimal solutions. We first formalize the notion of a vertex — both geometrically and algebraically.

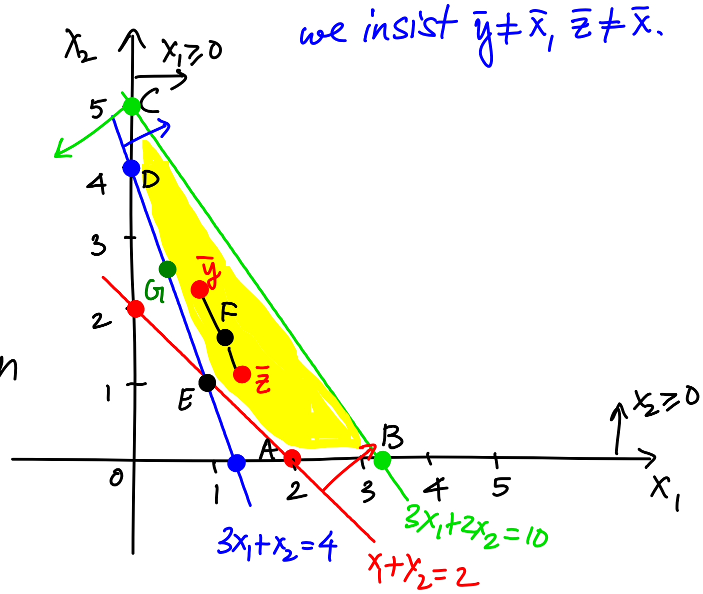
Vertices, Extreme points, and Basic Feasible Solutions

We are given a polyhedron $P = \{\bar{x} \in \mathbb{R}^n \mid A\bar{x} \geq \bar{b}\}$.

Def $\bar{x} \in P$ is an **extreme point** of P if there do not exist $\bar{y}, \bar{z} \in P$, $\bar{y} \neq \bar{x}$, $\bar{z} \neq \bar{x}$, such that $\bar{x} = \lambda\bar{y} + (1-\lambda)\bar{z}$ for some $\lambda \in [0,1]$.

λ could be 0 or 1, but we insist $\bar{y} \neq \bar{x}$, $\bar{z} \neq \bar{x}$.

In words, you cannot write an extreme point as a convex combination of two other distinct points.



E is an extreme point, but F and G are not. For instance, G can be written as a convex combination of D and E.