# MATH 464 – Lecture 8 (02/02/2023)

Today: * vertex, basic feasible solution (bfs)
* extreme point $\Leftrightarrow$ vertex $\Leftrightarrow$ bfs

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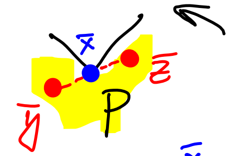
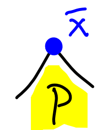
**Recall:** $\bar{x} \in P$ is an **extreme point** if $\nexists\ \bar{y}, \bar{z} \in P$, $\bar{y} \neq \bar{x}$, $\bar{z} \neq \bar{x}$, s.t. $\bar{x} = \lambda \bar{y} + (1-\lambda)\bar{z}$ for some $\lambda \in [0,1]$.

P being convex is crucial here.

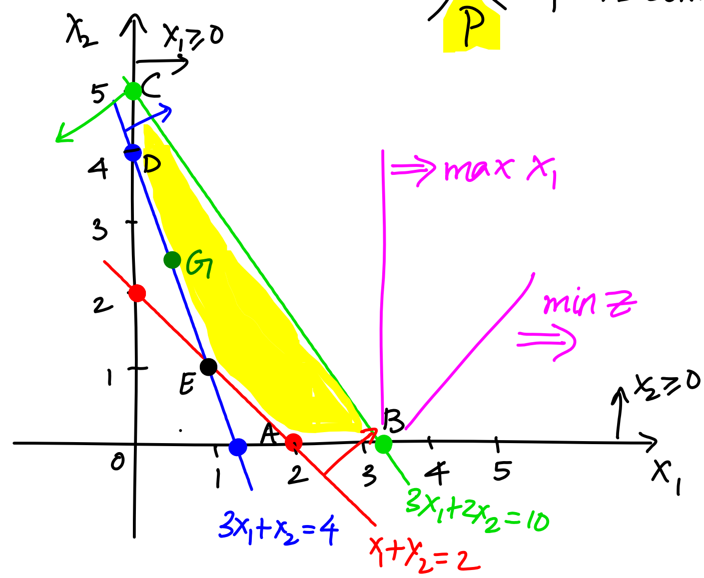



P not convex




P is convex

$$\begin{aligned}
\min \quad & 2x_1 + x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 2 \\
& 3x_1 + x_2 \geq 4 \\
& 3x_1 + 2x_2 \leq 10 \\
& x_1, x_2 \geq 0
\end{aligned}$$




**Def** $\bar{x} \in P$ is a **vertex** of $P$ if there is a vector $\bar{c} \in \mathbb{R}^n$ such that $\bar{c}^T\bar{x} < \bar{c}^T\bar{y}$ $\forall\, \bar{y} \in P$, $\bar{y} \neq \bar{x}$.

So, $\bar{x}$ is the unique optimal solution of some LP with P as the feasible region.

max $x_1 \equiv$ min $-x_1$

E is a vertex, $\bar{c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ works. Similarly B is a vertex, and $\bar{c} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ works for B. But G is not a vertex – no $\bar{c}$ has G as the unique optimal solution. Recall that $\bar{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ has $\overline{DE}$ (the entire line segment) as the set of optimal solutions.

There could be many $\bar{c}$ for which B is the unique optimal solution to min $\bar{c}^T\bar{x}$.

The above two definitions (extreme point and vertex) are geometric, and hence intuitive. But we need an equivalent definition that is algebraic, so that we could do computations.

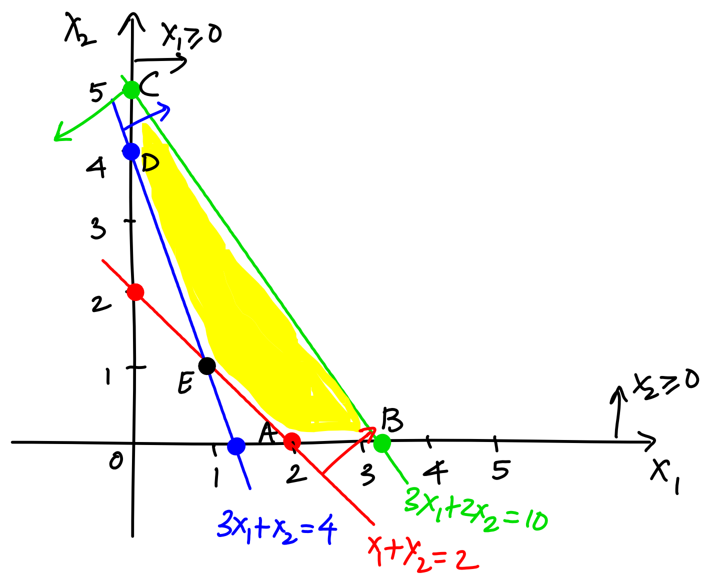
We need a few more concepts first.

## Active/binding Constraint

**Def** A constraint $\bar{a}_i^T \bar{x} \geq b_i$ is called **binding** or **active** at $\bar{x}^* \in P$ if $\bar{a}_i^T \bar{x}^* = b_i$, i.e., the constraint is satisfied as an equation.

Note: Equality constraints are always active (at all $\bar{x} \in P$).

$$\begin{aligned} \min \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 3x_1 + x_2 \geq 4 \\ & 3x_1 + 2x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$




Thus, $3x_1 + 2x_2 \leq 10$ is active at B, but not active at A or E.

Notice that at each vertex A, B, C, D, and E, two constraints are active. For instance, at B, $3x_1 + 2x_2 \leq 10$ and $x_2 \geq 0$ are binding. At D, $3x_1 + x_2 \geq 4$ and $x_1 \geq 0$ are active. At E, $x_1 + x_2 \geq 2$ and $3x_1 + x_2 \geq 4$ are active.

We formalize this intuition to higher dimensions. If there are $n$ constraints active at $\bar{x}^*$, then $\bar{x}^*$ is a solution to a system of those $n$ linear equations. If the vectors $\bar{a}_i$ defining these $n$ constraints are LI, then $\bar{x}^*$ is the unique solution to this linear system. We formalize these concepts in the following definition.

**Def** Let $P$ be a polyhedron with equality and inequality constraints.

(a) $\bar{x}^* \in \mathbb{R}^n$ is a **basic solution** of $P$ if — $\bar{x}^*$ need not be feasible here!

(i) all equality constraints are active; and

(ii) out of all constraints active at $\bar{x}^*$, there are $n$ of them which are LI. → If the constraints are $\bar{a}_i^T \bar{x} = b_i$ for $i \in I$, then $\{\bar{a}_i\}_{i \in I}$ is LI.

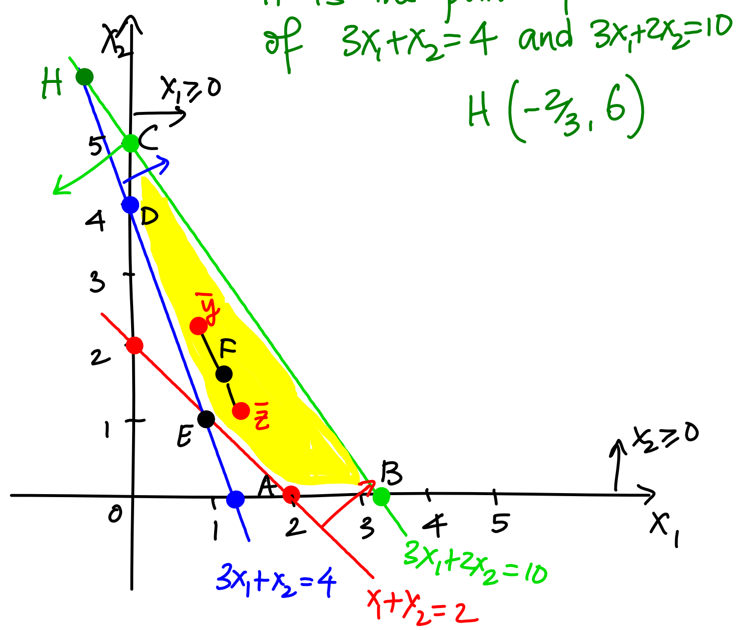
(b) If $\bar{x}^*$ is a basic solution that satisfies all constraints of $P$, then it is a **basic feasible solution** (**bfs**).

H is the point of intersection of $3x_1 + x_2 = 4$ and $3x_1 + 2x_2 = 10$

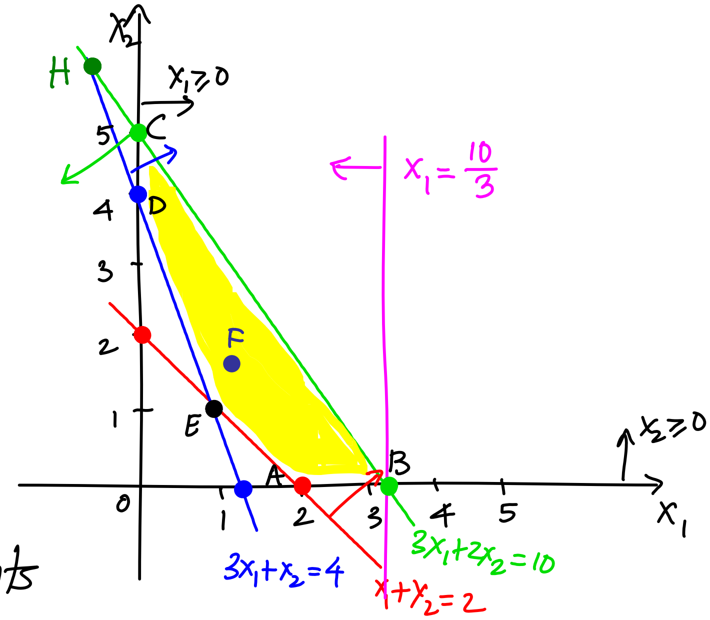
$H(-2/3, 6)$

H is a basic solution, but not a bfs

A, B, C, D, E are bfs's.



Notice that $F$ is not a basic solution, even though it is feasible, since none of the constraints are active at $F$.




Now we add another constraint to the original set of 3 constraints (and nonnegativity)

$$
\begin{aligned}
x_1+x_2 &\geq 2\\
3x_1+x_2 &\geq 4\\
3x_1+2x_2 &\leq 10\\
x_1 &\leq \frac{10}{3}\\
x_1, x_2 &\geq 0
\end{aligned}
$$

Now, three constraints are active at $B$: $3x_1+2x_2 \leq 10$, $x_1 \leq \frac{10}{3}$, $x_2 \geq 0$. Any subset of two of these constraints is LI. Equivalently, with $\underset{3x_1+2x_2\leq 10}{\bar{a}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$, $\underset{x_1 \leq \frac{10}{3}}{\bar{a}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$, $\underset{x_2 \geq 0}{\bar{a}_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$,

**any two** of $\{\bar{a}_3, \bar{a}_4, \bar{a}_6\}$ are LI. So $B$ is still a bfs

$\bar{a}_i$ is the coefficient vector of the $i^{th}$ constraint: $\bar{a}_i^T \bar{x} \geq b_i$.

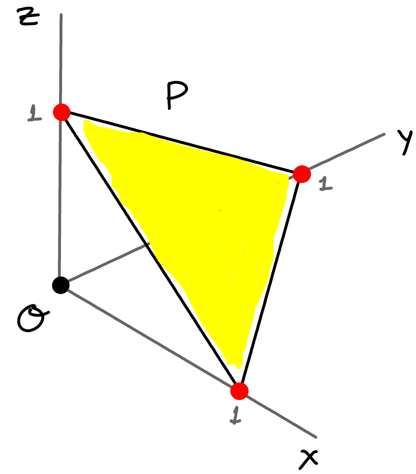
$x_1 \geq 0$, $x_2 \geq 0$ are the $5^{th}$ and $6^{th}$ constraints, after adding $x_1 \leq 10/3$ as the $4^{th}$ constraint.

Recall definition of bfs; in particular to requirement (a) i.

$$P = \{ \bar{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x_j \geq 0\ \forall j \}.$$

Note that $O$ (the origin) has 3 LI constraints active ($x_j \geq 0\ \forall j$).

But $O$ is not a basic solution, as it does not satisfy the equality constraint.




But if we write

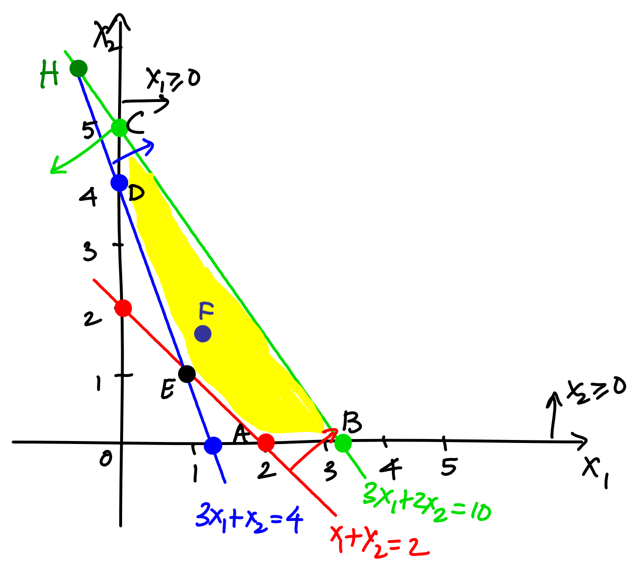
$$P = \left\{ \bar{x} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 + x_2 + x_3 \leq 1 \\ x_1 + x_2 + x_3 \geq 1 \end{array},\ x_j \geq 0\ \forall j \right\},$$

$O$ is a basic solution now! → Hence, the way we present inequalities matter!

But there is no such confusion for bfs's!

Back to our example:

At $D(0,4)$, $3x_1 + x_2 \geq 4$ and $x_1 \geq 0$ are active, and are LI.



**Theorem 2.3 (BT-ILO)** Extreme point $\Leftrightarrow$ vertex $\Leftrightarrow$ bfs.

see BT-ILO!

Or, the definitions are equivalent.

The proof is presented for two implications here.

**Proof** 1. (vertex $\Rightarrow$ extreme point)

Let $\bar{x}^*$ be a vertex of $P$. Hence there exists $\bar{c} \in \mathbb{R}^n$ such that $\bar{c}^T\bar{x}^* < \bar{c}^T\bar{y}$ $\forall \bar{y} \in P$, $\bar{y} \neq \bar{x}^*$.

Let $\bar{y}, \bar{z} \in P$, $\bar{y} \neq \bar{x}^*$, $\bar{z} \neq \bar{x}^*$, and $\lambda \in (0,1)$. → we do not include the end points, as we want $\bar{y}, \bar{z} \neq \bar{x}^*$.

So we get $\bar{c}^T\bar{x}^* < \bar{c}^T(\underbrace{\lambda\bar{y} + (1-\lambda)\bar{z}}_{\text{is } \neq \bar{x}^* \text{ for any } \lambda \in (0,1).})$

$\Rightarrow$ $\bar{x}^* \neq \lambda\bar{y} + (1-\lambda)\bar{z}$, or we cannot write $\bar{x}^*$ as a convex combination of $\bar{y}$ and $\bar{z}$. So, $\bar{x}^*$ is an extreme point.

2. (extreme point $\Rightarrow$ bfs)

We prove, equivalently, that (not bfs) $\Rightarrow$ (not extreme point).

Contrapositive argument: $(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A)$ for logical statements A, B

$\equiv$ "equivalent"; $\neg$ "negation" or "not"

Let $\bar{x}^* \in P$ be not a bfs. Hence among all the active constraints at $\bar{x}^*$, there do not exist a set of $n$ LI constraints.

Let $\bar{a}_i^T\bar{x}^* = b_i$, $i \in I$ be the set of constraints active at $\bar{x}^*$.

Hence, $\{\bar{a}_i \in \mathbb{R}^n, i \in I\}$ are not LI. ($I$ → index set)

Hence, there exists some $\bar{d} \in \mathbb{R}^n$, $\bar{d} \neq \bar{0}$, such that $\bar{a}_i^T \bar{d} = 0 \ \forall i \in I$.

Recall from linear algebra that if the columns of $A = [\bar{a}_1 \cdots \bar{a}_k]$ are not LI, then the homogeneous system $A\bar{x} = \bar{0}$ has a non-trivial solution.

Let $\bar{y} = \bar{x}^* + \epsilon \bar{d}$ and $\bar{z} = \bar{x}^* - \epsilon \bar{d}$ for $\epsilon > 0$, but small.

We want to show that $\bar{y}, \bar{z} \in P$ for $\epsilon$ small enough. Since $\bar{x}^* = \frac{1}{2}(\bar{y} + \bar{z})$, that will certify that $\bar{x}^*$ is <u>not</u> an extreme point.

We get $\bar{a}_i^T \bar{y} = b_i \ \forall i \in I$ as $\bar{a}_i^T (\bar{x}^* + \epsilon \bar{d}) = b_i + \epsilon 0 = b_i$.

We'll finish the proof in the next lecture...