MATH 524: Lecture 11 (09/23/2025)

Today:
* 0-dimensional homology
* reduced homology
* relative homology

Recall

Theorem 7.1 [M] The group $H_0(K)$ of simplicial complex K is free abelian. If $\{v_\alpha\}$ is a collection of vertices such that there is one vertex from each connected component of $|K|$, then $\{v_\alpha\}$ is a basis for $H_0(K)$.

Proof (ideas)

Step 1

0-skeleton, i.e., vertices

homologous

(i) For $v, w \in K^{(0)}$, we define $v \sim w$ if there is a sequence $a_0, \ldots, a_n$, with $a_i \in K^{(0)}$ such that $a_0 = v$, $a_n = w$, and $(a_i, a_{i+1}) \in K^{(1)}\ \forall i$.

the orientation does not matter; we just need (a_i, a_{i+1}) as an edge.

We also define $C_v = \bigcup \{ \operatorname{St} w \mid w \sim v \}$.

(ii) Show C_v is path-connected $\forall v \in K^{(0)}$.

(iii) If $C_v \neq C_{v'}$ (i.e., are distinct), then they are disjoint.

It follows that C_v are the connected components of $|K|$.

$\square$

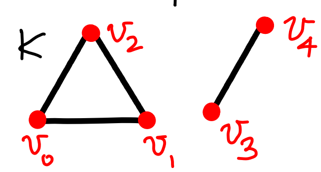
Step 2 Let $\{v_\alpha\}$ be a collection of vertices of K with one vertex from each component. Each 0-chain in a single component is homologous to v_α. Hence every 0-chain on K is homologous to a linear combination of elementary 0-chains v_α.

Let $\bar{c} = \sum n_\alpha v_\alpha$ be a general 0-chain. Suppose $\bar{c} = \partial \bar{d}$ for 1-chain. We can write $\bar{d} = \sum \bar{d}_\alpha$ where $\bar{d}_\alpha$ has terms of $\bar{d}$ carried by C_α. Consider one such component: $\partial \bar{d}_\alpha \sim n_\alpha v_\alpha$. It follows that $n_\alpha = 0$ here.
Why?

Let $\varepsilon : C_0(K) \to \mathbb{Z}$ be the homomorphism defined by $\varepsilon(v) = 1 \ \forall v \in K^{(0)}$.
Then $\varepsilon(\partial[v,w]) = \varepsilon(w-v) = 1-1 = 0$. Hence $0 = \varepsilon(\partial \bar{d}_\alpha) = \varepsilon(n_\alpha v_\alpha) = n_\alpha$. $\square$

Note: $\beta_0 = \operatorname{rk}(H_0(K))$ counts the number of connected components

Another example: $\beta_0(K) = 2$ here $\rightarrow$
$\{v_0, v_3\}$ is a basis for $H_0(K)$



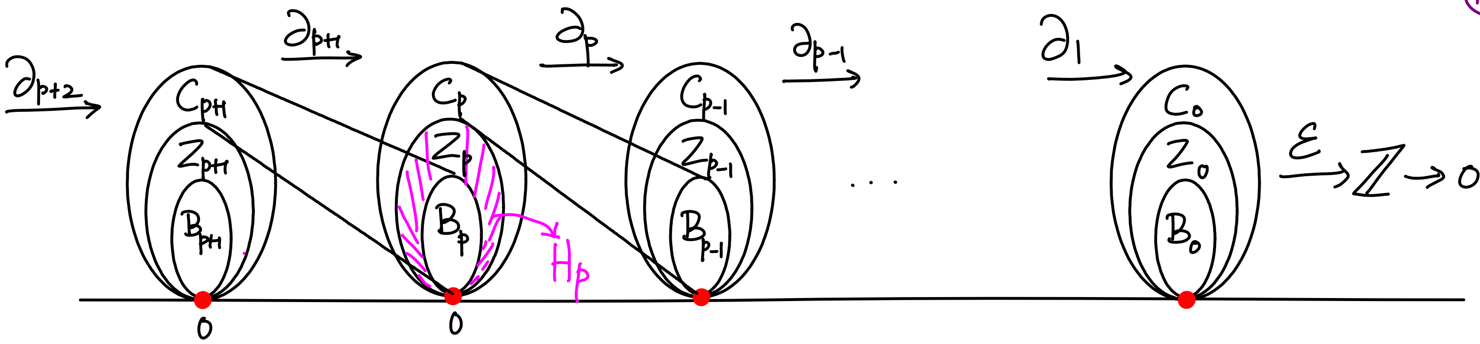
To follow the intuition for $p \geq 1$ of $\beta_p = 1$ when 1 $(p+1)$-dimensional "patch" is missing (thus creating a p-dim hole), we want $\beta_0 = 1$ when 1 edge, for instance, is missing, i.e., when there are two components (not 1). To this end, we define reduced homology groups.

Reduced Homology Groups

Let $\varepsilon : C_0(K) \to \mathbb{Z}$ be a surjective homomorphism defined by $\varepsilon(v) = 1 \ \forall v \in K^{(0)}$. For a 0-chain $\bar{c}$, $\varepsilon(\bar{c})$ is the sum of the values of $\bar{c}$ on vertices of K. ε is the **augmenting map** for $C_0(K)$. Also, $\varepsilon(\partial_1 \bar{d}) = 0$ for all 1-chains $\bar{d}$. So we define the **reduced homology group** of K in dimension 0 as

$$\tilde{H}_0(K) = \ker \varepsilon / \operatorname{im} \partial_1 .$$

Also, if $p > 0$, $\tilde{H}_p(K) = H_p(K)$.



Theorem 7.2 [M] $\tilde{H}_0(K)$ is free abelian, and $\tilde{H}_0(K) \oplus \mathbb{Z} \simeq H_0(K)$.

So, $\tilde{H}_0(K)$ vanishes if K is connected. Else $\{v_\alpha - v_{\alpha_0}\}$ for $\alpha \neq \alpha_0$ form a basis for $\tilde{H}_0(K)$. Here v_{α_0} is any one of the v_α's, which are from each connected component.

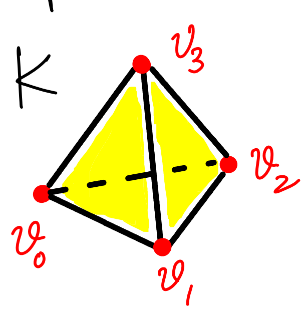
Proof If $\bar{c} \in \ker \varepsilon$, then $\varepsilon(\bar{c}) = \varepsilon(\bar{c}') = 0$, where $\bar{c}' \sim \bar{c}$ and $\bar{c}' = \sum n_\alpha v_\alpha$. But $\varepsilon(\bar{c}') = \sum n_\alpha \varepsilon(v_\alpha) = \sum n_\alpha$.

If $|K|$ has only one component, $\bar{c}' = 0$. If $|K|$ has more than one component, then $\bar{c}'$ is a linear combination of $\{v_\alpha - v_{\alpha_0}\}$.

We refer to $rk(\tilde{H}_0(K)) = \tilde{\beta}_0$ as the reduced 0-th betti number of K.

We get $\tilde{\beta}_0 = \beta_0 - 1$ and $\tilde{\beta}_p = \beta_p \ \forall\, p \geq 1$.

Homology of a p-simplex



solid tetrahedron

K: σ (3-simplex) and all its faces.

$\tilde{H}_i(K) = 0 \ \forall\, i$.

$\tilde{H}_3(K) = 0$, $\tilde{H}_2(K) = 0$, $\tilde{H}_1(K) = 0$, but $H_0(K) \simeq \mathbb{Z}$, and hence $\tilde{H}_0(K) = 0$.

Let Σ^{p-1} be the simplicial complex whose polytope is $\operatorname{Bd}\sigma$.

Then, $\tilde{H}_i(\Sigma^{p-1}) = 0$ for $i \neq p-1$, and $\tilde{H}_{p-1}(\Sigma^{p-1}) \simeq \mathbb{Z}$.

Here (for $p=3$), Σ^2 consists of the four triangles that are faces of σ, and their own faces. There are no tetrahedra in Σ^2, so $\bar{r} = n\sum_{i=0}^{3}\tau_i$, where τ_i are the triangles, is a 2-cycle which is not a 2-boundary for each $n \in \mathbb{Z}$, $n \neq 0$. Hence $\{\bar{r}\}$ is a basis for $\tilde{H}_2(\Sigma^2)$.

Relative Homology

We often want to talk about homology groups restricted to some parts of the given simplicial complex K. In particular, we want to "avoid" a subcomplex from consideration. Given a subcomplex K_0, a chain carried by K_0 is trivially extended to a chain in all of K by assigning zero as the coefficient for all simplices in K but not in K_0. Intuitively, we want to "zero out" all chains in K_0, and talk about homology groups in K modulo K_0.

Def If $K_0 \subseteq K$ is a subcomplex, the quotient group $C_p(K)/C_p(K_0)$ is the group of **relative p-chains of K modulo K_0**, denoted $C_p(K, K_0)$.

Notice that $C_p(K_0)$ can be naturally considered as a subgroup of $C_p(K)$, by assigning coefficients of zero to simplices not in K_0.

$C_p(K,K_0)$ is free abelian, and has as a basis all cosets of the form

$$\{\sigma_i\} = \sigma_i + C_p(K_0)$$

where σ_i is a p-simplex of K not in K_0.

Intuitively, adding any chain from $C_p(K_0)$ to σ_i is like adding "zero" as far as $C_p(K,K_0)$ is concerned.

the "absolute" boundary operator ∂

$\partial : C_p(K_0) \longrightarrow C_{p-1}(K_0)$ is just the restriction of ∂ on $C_p(K)$ to K_0. This homomorphism induces a homomorphism

$$\partial : C_p(K,K_0) \longrightarrow C_{p-1}(K,K_0).$$

We will use ∂ to denote both the absolute and the relative boundary operators

This is the **relative boundary operator**.

Like the absolute boundary operator, the relative boundary operator also satisfies $\partial \circ \partial = 0$. We let

$$Z_p(K,K_0) = \ker \partial_p : C_p(K,K_0) \to C_{p-1}(K,K_0),$$

$$B_p(K,K_0) = \operatorname{im} \partial_{p+1} : C_{p+1}(K,K_0) \longrightarrow C_p(K,K_0), \text{ and}$$

$$H_p(K,K_0) = Z_p(K,K_0) / B_p(K,K_0).$$

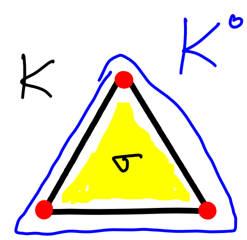
These groups are called the **relative p-cycle, relative p-boundary**, and the **relative homology group** of dimension p of K modulo K_0.

A relative p-chain $\bar{c}$ is a relative p-cycle iff $\partial_p \bar{c}$ is carried by K_0. Furthermore, it's a relative p-boundary iff there exists a $(p+1)$-chain $\bar{d}$ of K such that $\bar{c} - \partial_{p+1}\bar{d}$ is carried by K_0.

Recall that in the absolute case, we wanted $\partial_p \bar{c}$ and $\bar{c} - \partial_{p+1}\bar{d}$ to be empty, respectively.

Example 1 Let K be the p-simplex σ and all its faces. Let K_0 be $K^{(p-1)}$, i.e., the simplicial complex made of all proper faces of σ. Then

$H_i(K, K_0) = 0 \;\; \forall\, i \neq p$, and



$H_p(K, K_0) \simeq \mathbb{Z}$, and $\{\sigma\}$ is a basis.

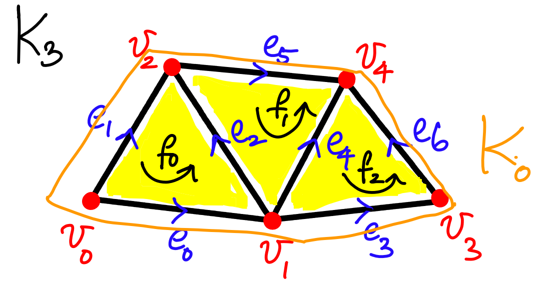
$H_2(K, K_0) \simeq \mathbb{Z}$

as $\partial_p \sigma$ is carried by K_0, and hence it's a relative p-cycle. There are no $(p+1)$-simplices to bound, so it's not a relative p-boundary.

Notice that $\bar{c} = n\sigma$ for $n \in \mathbb{Z}$ is not an absolute cycle, as $\partial_2 \bar{c} \neq 0$.

Example 2 (Example 3 in Lecture 9)

Let K_0 be the subcomplex consisting of e_0, e_1, e_3, e_5, e_6 and all the vertices. Then we get



$H_2(K_3, K_0) \simeq \mathbb{Z}$, and $\bar{r} = \sum_{i=0}^{2} f_i$ is a generator.

We use the same techniques as before. The triangles are oriented CCW. Then $\bar{r}$, the 2-chain which is the sum of the triangles taken with multipliers of 1 each, has $\partial \bar{r}$ carried by K_0. Hence it is a relative 2-cycle.

There are no tetrahedra in K_1 and hence there are no 2 boundaries (absolute or relative). Hence $\bar{r}$ generates $H_2(K_1, K_0)$.

We now consider $H_1(K, K_0)$. Using the same "pushing off edges in the middle" argument as before, we get that any 1-chain in K is homologous to a 1-chain carried by K_0, and hence is a relative 1-cycle that is trivial. In more detail, every 1-chain in K not in K_0 is a relative 1-cycle, and is also a relative 1-boundary since we can find a 2-chain generated by f_1 and f_2 whose difference with this 1-chain is carried by K_0.

Thus, $H_1(K_3, K_0) = 0$, as any 1-chain in K is homologous to a 1-chain carried by K_0.

Here, $H_0(K_3, K_0) = 0$ as well, as $v_i \in K_0 \ \forall i$.

Notice the similarity between Examples 1 (for $p=2$) and 2 – the homology groups are the same. Also, notice that $|K_3|$ and $|K|$ are homeomorphic (both are discs), and $|K_0|$'s are also homeomorphic (to a circle in each case). These examples seem to indicate that relative homology groups are determined by the underlying space, and not by the choice of the simplicial complexes – indeed, this is true in general, but the proof is technical.