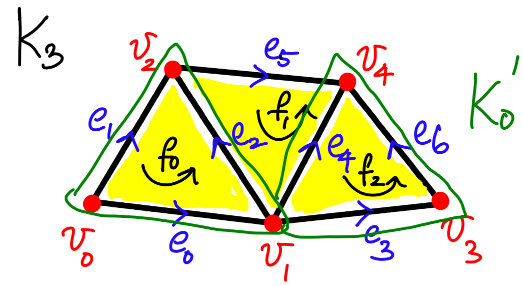
MATH 524: Lecture 12 (09/25/2025)

Today: * More examples of relative homology
* Excision theorem

Example 2 (continued..)



Now, consider K_0' as the subcomplex made of $\{e_0, e_1, e_2, e_3, e_4, e_6\}$, and all vertices.

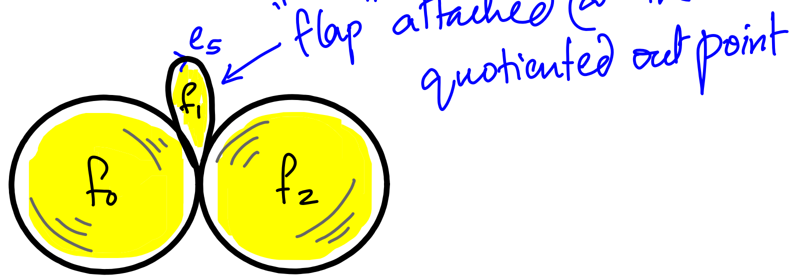
$H_2(K_3, K_0') \simeq \mathbb{Z} \oplus \mathbb{Z}$ here!

If $\bar{r}' = n_0 \bar{f}_0 + n_2 \bar{f}_2$, then $\partial \bar{r}'$ is carried by K_0, and hence is a relative 2-cycle. And $n_0, n_2 \in \mathbb{Z}$ could be chosen arbitrarily. Indeed, $\{\bar{f}_0, \bar{f}_1\}$ is a basis.

But $H_1(K_3, K_0') = 0$ still. All relative 1-chains are generated by $\{\bar{e}_5\}$, which happens to be a relative 1-cycle as $\partial \bar{e}_5$ is carried by K_0'. But $\bar{e}_5$ is also a relative 1-boundary as $\bar{e}_5 + \partial_2 \bar{f}_1$ is carried by K_0'.

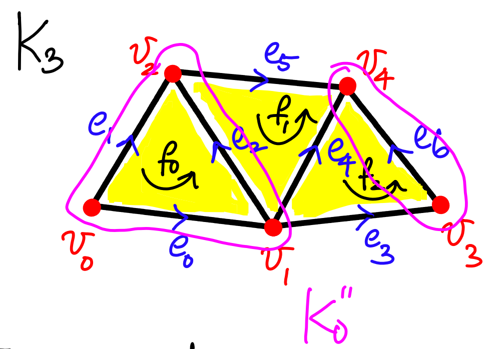
Similarly, $H_0(K_3, K_0') = 0$, as all $v_i \in K_0'$.

Intuitively, one could think of K_3/K_0' as comprised of two spheres touching each other at a point, along with a "flap" (disc) attached to the same point of contact between the spheres.



Now consider K_0'' as shown:

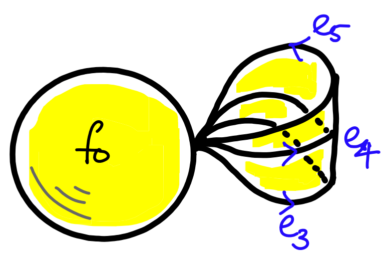
K_0'' : $\{e_0, e_1, e_2, e_6\}$ and all vertices.



We get that $H_2(K_3, K_0'') \simeq \mathbb{Z}$, and $\{\bar{f}_0\}$ is a basis. Notice that $n_1\bar{f}_1 + n_2\bar{f}_2$ is not a relative 2-cycle for any $n_1, n_2 \in \mathbb{Z}$, except $n_1 = n_2 = 0$.

$H_1(K_3, K_0'') \simeq \mathbb{Z}$. We can push off any relative 1-chain in K_3/K_0'' of $\bar{e}_3$ and $\bar{e}_4$, for instance, leaving $\bar{e}_5$ as a generator of $H_1(K_3, K_0'')$.

Intuitively, one could imagine "shrinking" all of $|K_0''|$ to a point, and consider homology of K modulo that point. In this sense, one could think of

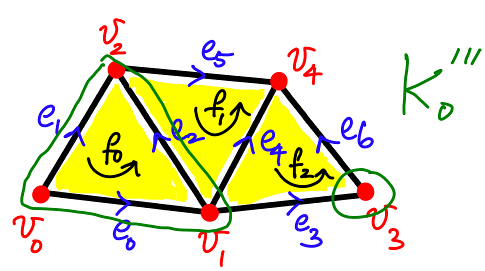


Also, notice that different choices of K_0 lead to different $H_p(K, K_0)$ groups.

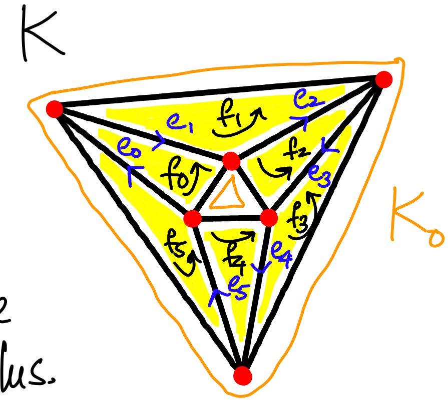
Now consider K_0''' as shown.

What is $H_0(K_3, K_0''') = ?$

Think! Think!



Example 3 (Annulus)



Let K consist of the six triangles $f_0, \dots, f_5$ as shown here, with the triangle in the middle missing. Hence $|K|$ is homeomorphic to the 2D annulus.

Let K_0 consist of all the boundary edges and their vertices, i.e., both the inner and outer circular boundaries.

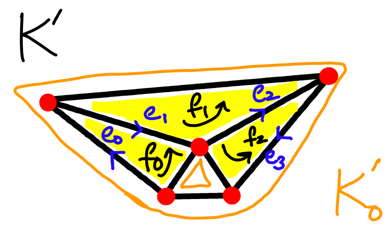
Then $H_2(K, K_0) \simeq \mathbb{Z}$. Notice that $\bar{r} = \sum_{i=0}^{5} f_i$ has $\partial_2 \bar{r}$ carried by K_0. Indeed, $\bar{r}$ generates $H_2(K, K_0)$.

What about $H_1(K, K_0)$? Notice that we can push any relative 1-chain off of $\bar{e}_1$ using $\bar{f}_1$, and then $\bar{e}_2$ using $\bar{f}_2$, and so on, all the way around. But we will be left with $\bar{e}_0$ in this case. Thus, $\{\bar{e}_0\}$ is a relative 1-cycle which is not a relative 1-boundary. Thus, $H_1(K, K_0) \simeq \mathbb{Z}$.

But now consider a modified complex as shown here.

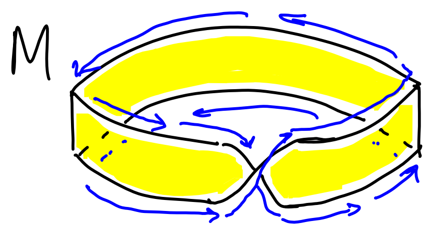
Notice that e_0 is carried by K_0'.

Indeed, $H_1(K', K_0') = 0$ here.



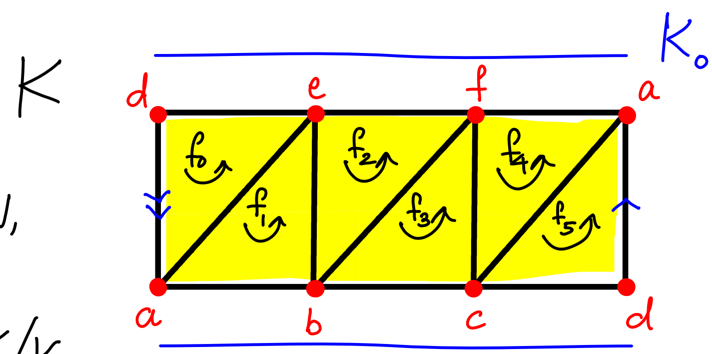
Example 4 Torsion in relative homology groups of Möbius strip:

Recall:



$H_1(M) \simeq \mathbb{Z}$ no torsion! despite the twist.

Let K triangulate the Möbius strip, as shown. With all triangles oriented ccw, notice that for the 2-chain $\bar{r} = \sum_{i=0}^{5} f_i$, $\partial_2 \bar{r} = 2(\overline{da})$ in K/K_0.



$\partial_2 \bar{r}$ is $2\,\overline{da}$ + edges in K_0

Let K_0 be the "edge" of the Möbius strip, as shown. Then $H_1(K, K_0) \simeq \mathbb{Z}_2$, as $2(\overline{da})$ is a relative boundary, but $(\overline{da})$ is not. Of course, $\overline{da}$ is a relative 1-cycle here.

Note that every edge "going across" is a relative 1-cycle here, e.g., $\overline{ae}$, $\overline{bf}$, $\overline{ca}$, etc.

Intuitively, we can "shrink" all of K_0 to a point, and consider $|K|/|K_0|$, after this reduction. This point of view affords some powerful applications/tools. In particular, we could make changes to the interior of K_0, without affecting $H_p(K, K_0)$. We make this notion precise in the following theorem.

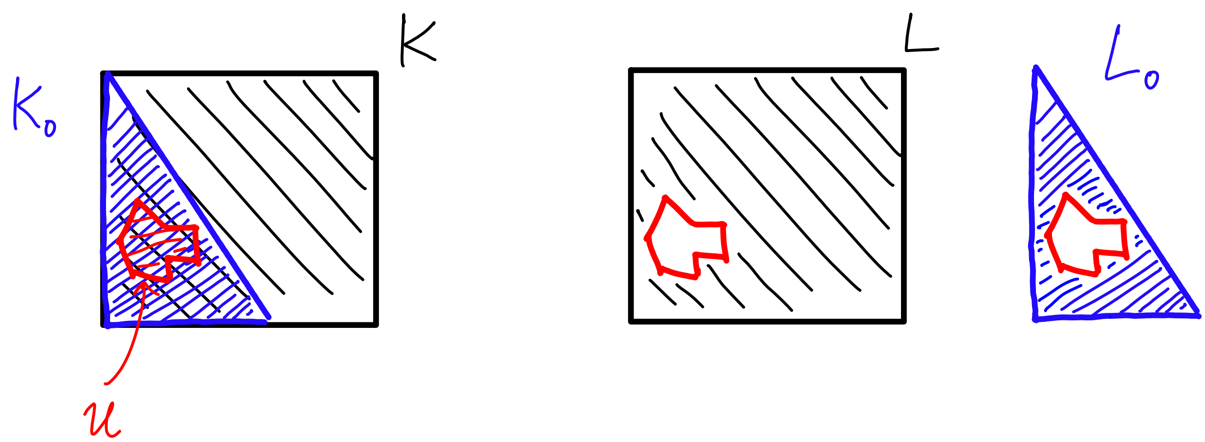
Theorem 9.1 [M] (Excision theorem) Let K_0 be a subcomplex of K.

strict subset

Let $\mathcal{U} \subset |K_0|$ be an open set such that $|K| - \mathcal{U}$ is the polytope of a subcomplex L of K and let L_0 be the subcomplex whose polytope is $|K_0| - \mathcal{U}$. Then inclusion induces an isomorphism

$$H_p(L, L_0) \simeq H_p(K, K_0).$$

Here is a schematic illustration. The spaces here are supposed to represent simplicial complexes.



In many cases, L/L_0 is much nicer, or easier to compute with, than K/K_0. In particular, if $\mathcal{U}$ is chosen to be large (but still contained in K_0), L and L_0 might be much simpler than K and K_0. We will encounter applications of the excision theorem later on...

Proof idea: Consider the composite map ϕ

$$C_p(L) \to C_p(K) \longrightarrow C_p(K)/C_p(K_0)$$

defined as inclusion followed by projection. "project out" parts in K_0

a p-chain in L is extended to a p-chain in K by setting the weights on p-simplices in K/L to zero.

ϕ is surjective, as $C_p(K)/C_p(K_0)$ has as basis all cosets $\{\sigma_i\}$ for p-simplices σ_i in K not in K_0, and all such $\sigma_i \in L$. Also, $\ker \phi$ is $C_p(L_0)$.

So, ϕ induces an isomorphism $C_p(L)/C_p(L_0) \simeq C_p(K)/C_p(K_0) \;\forall p$.

And ∂ is preserved under this isomorphism.

The p-simplex σ is mapped to empty (i.e., to zero) if it is in L_0 by the projection part of ϕ.

Hence, $H_p(L, L_0) \simeq H_p(K, K_0)$. $\square$

We now turn to simplicial maps, and how the groups we have studied – chains, cycles, boundaries, and homology groups – behave under them. We introduce several useful algebraic tools in this process.

Homomorphisms induced by Simplicial Maps

§12 in [M]

Recall Simplicial map: Given simplicial complexes K and L, $f: K \to L$ is a simplicial map if f is a continuous map of $|K|$ to $|L|$ that maps each simplex of K linearly onto a simplex of L.

We could start with the corresponding vertex map, and extend the same linearly to the simplicial map.

Note that a simplex in K could be mapped to a lower dimensional simplex in L by f. We define a homomorphism from f by "staying in the same dimension."

$(\dim(f(\sigma)) \leq \dim \sigma)$.

Def Let $f: K \to L$ be a simplicial map. If $(v_0, \dots, v_p)$ is a simplex of K, then $f(v_0), \dots, f(v_p)$ span a simplex of L. We define a homomorphism $f_\#: C_p(K) \to C_p(L)$ by defining it on oriented p-simplices as follows.

$$f_\#([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_i) \text{ are distinct,} \\ 0, & \text{otherwise.} \end{cases}$$

This map is indeed well-defined, i.e., $f_\#(-\sigma) = -f_\#(\sigma)$. If we swap v_i and v_j in $[v_0, \dots, v_p]$, the sign of the right-hand side expression is changed.

The family of homomorphisms $\{f_\#\}$, one in each dimension, is called the **chain map induced by the simplicial map f**.