MATH 524: Lecture 13 (09/30/2025)

Today: simplicial maps and induced homomorphisms

Recall For $f: K \to L$, $f_{\#}([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)], & \text{if } f(v_i) \neq f(v_j), \\ 0 & \text{o.w.} \end{cases}$

Notation: Ideally, we should write $(f_{\#})_p : C_p(K) \to C_p(L)$, for each dimension p. But we avoid writing p, when the dimension is evident. We will work with $f_{\#}$ and ∂ in both K and L. We use ∂ to denote the boundary in K and in L. We could be particular, and write ∂_K and ∂_L, when necessary.

Notice that $f_{\#}$ is a homomorphism for C_p groups. What about the homology groups H_p? It turns out that $f_{\#}$ induces a homomorphism f_* from $H_p(K)$ to $H_p(L)$.

Lemma 12.1 [M] The homomorphism $f_{\#}$ commutes with ∂; therefore $f_{\#}$ induces a homomorphism $f_* : H_p(K) \to H_p(L)$.

$$\begin{tikzcd}
C_p(K) \arrow[r, "(f_{\#})_p"] \arrow[d, "\partial_K"'] & C_p(L) \arrow[d, "\partial_L"] \\
C_{p-1}(K) \arrow[r, "(f_{\#})_{p-1}"] & C_{p-1}(L)
\end{tikzcd}$$

composing the maps along either path gives the same result

To be exact, we say $\partial_L \circ (f_{\#})_p = (f_{\#})_{p-1} \circ \partial_K$, or just briefly, $\partial f_{\#} = f_{\#} \partial$.

Proof We first show that

$$\partial f_{\#}([v_0,\dots,v_p]) = f_{\#}(\underbrace{\partial [v_0,\dots,v_p]}_{\sum_{i=0}^{p}(-1)^i [v_0 \dots \hat{v}_i \dots v_p]}) \quad \text{———} \quad (*).$$

Let τ be the simplex spanned by $f(v_0),\dots,f(v_p)$. We consider three cases, based on the dimension of τ.

Case 1. $\dim(\tau) = p$. Here, $f(v_0),\dots,f(v_p)$ are distinct, and hence the result follows directly since $f_{\#}$ and ∂ are homomorphisms.

Case 2. $\dim(\tau) \le p-2$. three or more vertices are mapped to one vertex, or two or more pairs are identified.

LHS of $(*)$ vanishes, as $f(v_i)$ are not all distinct.

RHS of $(*)$ is also 0, as $\forall i$, two or more terms out of $f(v_0),\dots,f(v_{i-1}), f(v_{i+1}),\dots, f(v_p)$ are the same.

Case 3 $\dim(\tau) = p-1$. exactly one pair of v_i's are mapped to the same vertex in L

WLOG, assume vertices are ordered such that

$f(v_0) = f(v_1) \neq f(v_2) \neq \dots \neq f(v_p)$. $f(v_0)=f(v_1)$, while the remaining $f(v_i)$ are all distinct, and also distinct from $f(v_0)$

Again, LHS of $(*)$ vanishes by definion.

RHS of $(*)$ has only two nonzero terms:

$[f(v_1), f(v_2),\dots,f(v_p)]$ and $-[f(v_0), f(v_2),\dots,f(v_p)]$.

As $f(v_0)=f(v_1)$, these two terms cancel.

Further, $f_\#$ carries cycles to cycles and boundaries to boundaries.

Let $\bar{z} \in Z_p(K)$. Then $\partial(\bar{z}) = 0$. By Lemma, $\partial f_\#(\bar{z}) = f_\# \partial(\bar{z}) = 0$. So $f_\#(\bar{z})$ is a cycle, i.e., $f_\#(\bar{z}) \in Z_p(L)$.

Similarly, if $\bar{b} \in B_p(K)$, then $\bar{b} = \partial_{p+1}\bar{d}$ for $\bar{d} \in C_{p+1}(K)$.

But $\partial f_\#(\bar{d}) = f_\# \partial \bar{d} = f_\#(\bar{b})$, and hence $f_\#(\bar{b}) \in B_p(L)$.

Thus, $f_\#$ induces a homomorphism of the homology groups, $f_*: H_p(K) \to H_p(L)$. $\square$

We can naturally combine the homomorphisms induced by multiple simplicial maps, as the following theorem describes.

Theorem 12.2 [M] (a) Let $i: K \to K$ be the identity simplicial map. Then $i_*: H_p(K) \to H_p(K)$ is the identity homomorphism.

(b) Let $f: K \to L$ and $g: L \to M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$, i.e., the following diagram commutes.

$$\begin{tikzcd}
H_p(K) \arrow[rr, "(g\circ f)_*"] \arrow[dr, "f_*"'] & & H_p(M) \\
& H_p(L) \arrow[ur, "g_*"'] &
\end{tikzcd}$$

This theorem presents the **functorial** property of the induced homomorphism. Think of H_p as an operator that assigns to each simplicial complex an abelian group, and $*$ as another operator that assigns to each simplicial map of one complex to another, a homomorphism between the corresponding abelian groups.

We say that $(H_p, *)$ is a "functor" from the "category" of simplicial complexes and simplicial maps to the "category" of abelian groups and homomorphisms.

Intuitively, a "category" consists of a collection of sets (or "objects") along with maps between them. A functor assigns pairs of such structure, i.e., (object, map) pairs, in one category to those in the other category such that it "preserves the structure" of the category.

The ideas of commuting diagrams in particular, and functoriality in general, are used widely in algebraic topology. We will see more of these concepts in the upcoming lectures.

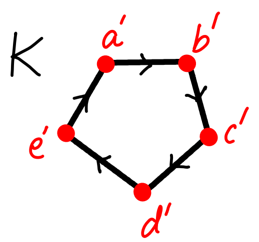
We study further the homomorphisms induced by simplicial maps. In particular, we talk about when distinct simplicial maps induce equal homomorphisms on homology groups.

Lemma 12.3 [M] $f_\#$ preserves the augmentation map ϵ; therefore it induces a homomorphism f_* of reduced homology groups.

Proof Let $f: K \to L$ be a simplicial map. Then $\epsilon f_\#(v) = 1$, and $\epsilon(v) = 1$ $\forall v \in K^{(0)}$. Hence $\epsilon \circ f_\# = \epsilon$. Thus $f_\#$ carries the kernel of $\epsilon_K : C_0(K) \to \mathbb{Z}$ into the kernel of $\epsilon_L : C_0(L) \to \mathbb{Z}$, and so it induces a homomorphism $f_* : \tilde{H}_0(K) \to \tilde{H}_0(L)$.

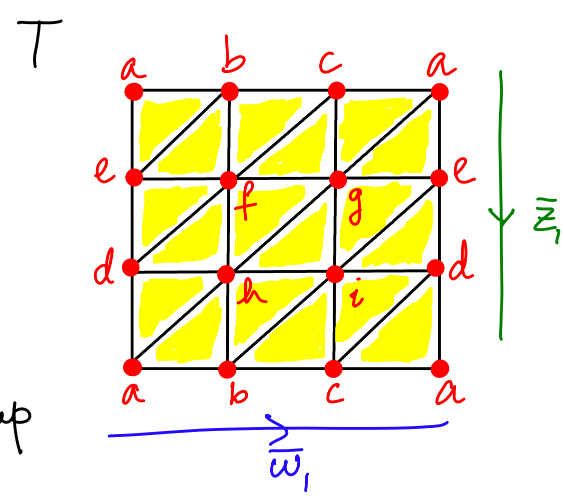
Example

Consider K that is a loop, and T, which represents $\mathbb{T}^2$.



$\bar{z}$: $a'b'c'd'e'$

$H_1(K) \simeq \mathbb{Z}$, $\{\bar{z}\}$ generates this group



As described previously, with $\bar{w}_1 = [a,b] + [b,c] + [c,a]$ and $\bar{z}_1 = [a,e] + [e,d] + [d,a]$, $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and $\{\bar{w}_1, \bar{z}_1\}$ is a basis.

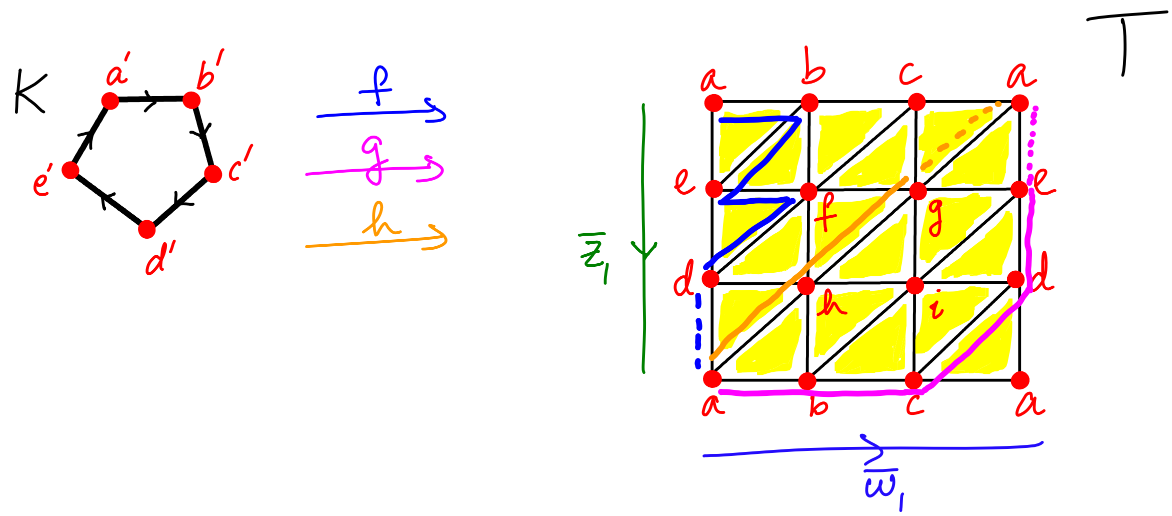
We consider three different simplicial maps $f, g, h : K \to T$, described by the maps for each vertex in K.

$f: a' \to a$
$b' \to b$
$c' \to e$
$d' \to f$
$e' \to d$

$g: a' \to a$
$b' \to b$
$c' \to c$
$d' \to d$
$e' \to e$

$h: a' \to a$
$b' \to h$
$c' \to h$
$d' \to g$
$e' \to g$

We can visualize the three maps as follows.

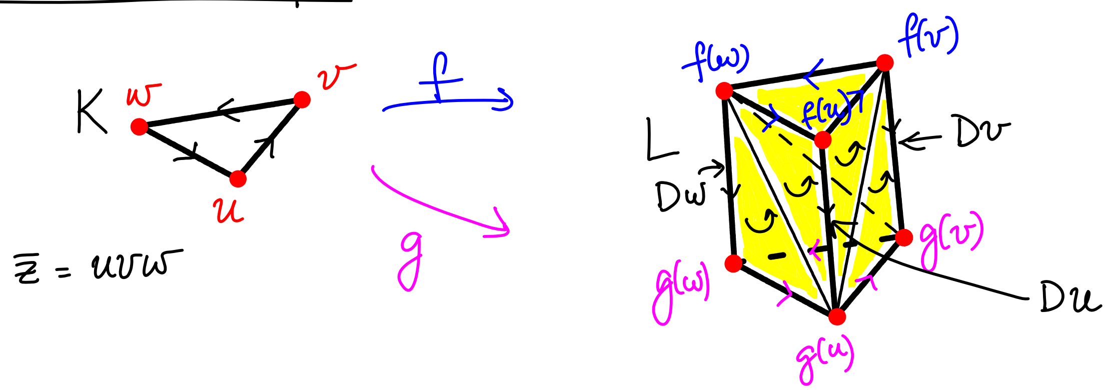


We can check that $f_\#(\bar{z}) \sim \bar{z}_1$, $g_\#(\bar{z}) \sim \bar{w}_1 - \bar{z}_1$, and $h_\#(\bar{z}) \sim \bar{w}_1 - \bar{z}_1$. Hence g_* and h_* are equal as homomorphisms of the first homology group.
It can be checked that g_* and h_* are equal as homomorphisms of the 0-dimensional homology groups as well.

When can this observation hold in general?

Given simplicial maps $f, g: K \to L$, we want to find conditions under which $f_\#(\bar{z}) \sim g_\#(\bar{z})$ $\forall \bar{z} \in Z_p(K)$. Thus we want to find a $(p+1)$-chain $D\bar{z}$ of L such that $f_\#(\bar{z}) - g_\#(\bar{z}) = \partial D\bar{z}$.

Here's an example where we can find $D\mathfrak{z}$ straightforwardly.



L consists of 6 triangles such that $|L|$ is the cylinder. K is made of 3 edges forming a cycle, which we term $\mathfrak{z}$. f and g are two simplicial maps which map $\mathfrak{z}$ to the top and bottom cycles, respectively, in L. The triangles in L can be oriented consistently, e.g., CCW when looking from outside.

Here, $D\mathfrak{z}$ can be chosen to be the 2-chain made of the 6 triangles in the middle.

But for a different pair of maps f' and g', it might not be as straightforward to identify $D\mathfrak{z}$ in all cases.

When can we find $D\mathfrak{z}$ easily for all cycles $\mathfrak{z}$? Could we specify some sufficient conditions for the existence of $D\mathfrak{z}$?