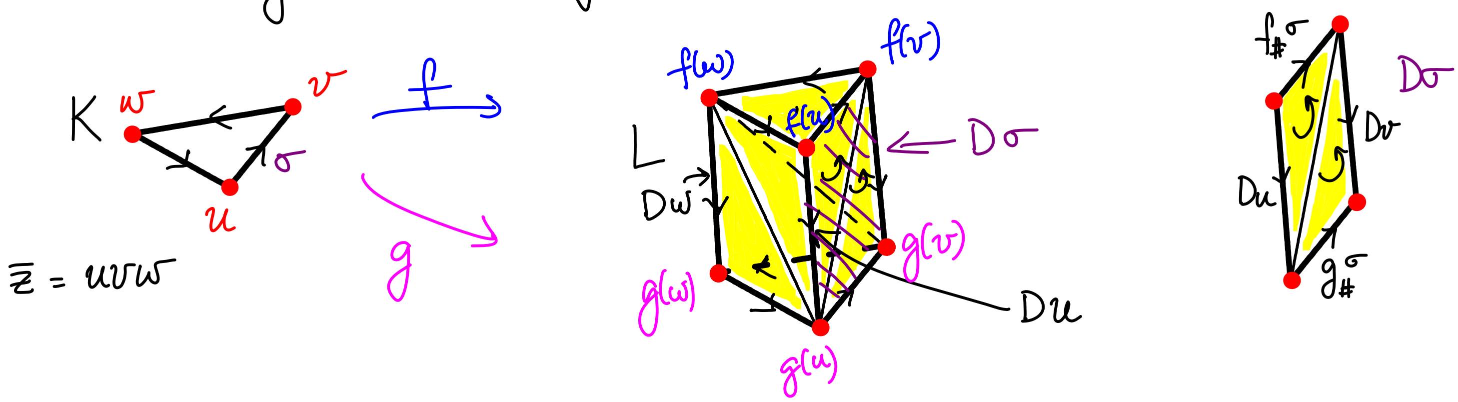
MATH 524: Lecture 14 (10/02/2025)

Today: * chain homotopy
* star condition

We continue with the example where we could identify $D\bar{z}$. But now we identify $D\sigma$ for elementary chains $\sigma \in K$, starting with vertices and proceeding to higher dimensional simplices. Our goal is to identify some sort of formula that $D\bar{c}$ should satisfy for a general p-chain $\bar{c} \in C_p(K)$.

For vertex $v \in K^{(0)}$, define Dv to be the edge in L connecting $f(v)$ and $g(v)$.



For edge uv, with $\sigma = uv$, $D\sigma$ is the sum of the two triangles between $(f(u), f(v))$ and $(g(u), g(v))$.

Notice that we get
$$\partial(D\sigma) = g_{\#}(\sigma) - Dv - f_{\#}(\sigma) + Du.$$

In other words, we have $\partial(D\sigma) = g_{\#}(\sigma) - f_{\#}(\sigma) - D(\underbrace{\partial\sigma}_{v-u})$.

This example in fact suggests the form that $D\sigma$ should satisfy in general. We want $D(\partial\sigma) + \partial(D\sigma) = g_{\#}(\sigma) - f_{\#}(\sigma)$.

We define the existence of such a $(p+1)$-chain for each p-simplex as the required sufficient condition in general for all p.

Def Let $f,g : K \to L$ be simplicial maps. Suppose that for all p, there is a homomorphism $D : C_p(K) \to C_{p+1}(L)$ which satisfies

$$\partial D + D \partial = g_\# - f_\# .$$

Then D is said to be a **chain homotopy** between $f_\#$ and $g_\#$.

Intuitively, the images of each p-simplex σ under f and g are "close" to each other if there is a chain-homotopy. Notice that the requirement is specified for all dimensions.

We could be more precise in writing the equation by including subscripts of dimension $(p, p+1)$ and simplicial complexes (K and L). We express the maps in detail as follows.

$$\begin{tikzcd}
 & C_{p+1}(L) \arrow[d, "(\partial_{p+1})_L"] \\
C_p(K) \arrow[ur, "D_p"] \arrow[r, shift left, "(f_\#)_p"] \arrow[r, shift right, "(g_\#)_p"'] \arrow[d, "(\partial_p)_K"'] & C_p(L) \\
C_{p-1}(K) \arrow[ur, "D_{p-1}"'] &
\end{tikzcd}$$

The detailed relation we want is the following:

$$(\partial_{p+1})_L D_p + D_{p-1} (\partial_p)_K = (g_\#)_p - (f_\#)_p .$$

But we usually will write $\partial D + D\partial = g_\# - f_\#$, for brevity.

The following theorem describes why we want to study chain homotopies.

Theorem 12.4 [M] If there is a chain homotopy between $f_\#$ and $g_\#$, then the induced homomorphisms f_* and g_*, for both reduced and absolute homology, are equal.

Proof If $\bar{z} \in Z_p(K)$, then

$$g_\#(\bar{z}) - f_\#(\bar{z}) = \partial D\bar{z} + D\partial\bar{z} = \partial D\bar{z} + 0.$$

So, $g_\#(\bar{z}) \sim f_\#(\bar{z})$, and hence $g_*(\{\bar{z}\}) = f_*(\{\bar{z}\})$.

We now give a sufficient condition for existence of a chain homotopy.

Def Two simplicial maps $f, g : K \to L$ are said to be **contiguous** if for every simplex $\sigma = (v_0 \dots v_p)$ of K, the points $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$ span a simplex τ of L.

Note: 1. $0 \le \dim(\tau) \le 2p+1$.
2. $f(\sigma)$ and $g(\sigma)$ are both faces of a (possibly) larger simplex τ of L.
i.e., $f(\sigma)$ and $g(\sigma)$ are "close" to each other

Theorem 12.5 [M] If $f, g : K \to L$ are contiguous simplicial maps, then a chain homotopy exists between $f_\#$ and $g_\#$.

Proof (outline; see [M] for details)

For $\sigma = v_0, \dots, v_p$ of K, let $L(\sigma)$ be the subcomplex of L made of the simplex spanned by $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$, and all its faces. We should have the following results.

(1) $L(\sigma)$ is nonempty, $\tilde{H}_i(L(\sigma)) = 0 \ \forall i$.

(2) If τ is a face of σ, then $L(\tau) \subset L(\sigma)$.

(3) for every oriented simplex σ, $f_\#(\sigma)$ and $g_\#(\sigma)$ are both carried by $L(\sigma)$.

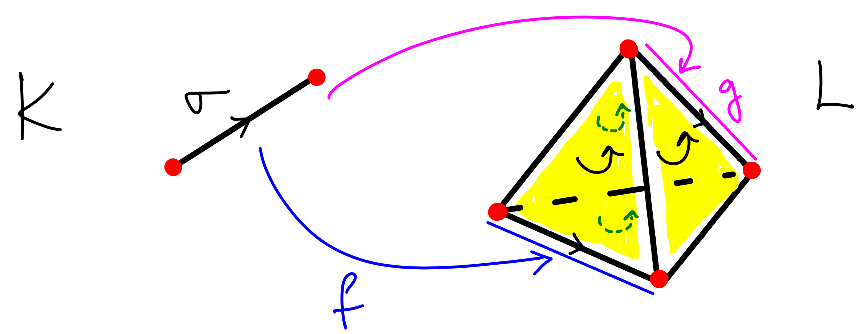
We now show that $D\sigma$ exists for each p-simplex σ using induction on p.

$p=0$ Notice that $\epsilon(g_\#(v) - f_\#(v)) = 1 - 1 = 0$. Hence $g_\#(v) - f_\#(v) \in \tilde{H}_0(L(v))$. But $\tilde{H}_0(L(v)) = 0$, so we can choose a 1-chain Dv of L carried by $L(v)$ such that

$$\partial(Dv) = g_\#(v) - f_\#(v).$$

(See [M] for the induction step going from $p-1$ to p). $\square$

Notice that the theorem guarantees the existence of some $D\sigma$ for each σ — the choice may not be unique. Indeed, consider the case where a 1-simplex σ gets mapped by f and g to two opposite edges of a tetrahedron. Then there are two choices for $D\sigma$ — the two triangles of the tetrahedron visible in front, or the other two tetrahedron lying behind.



Application to relative homology

Def Let $K_0 \subseteq K$ and $L_0 \subseteq L$ be subcomplexes. Let $f, g : (K, K_0) \to (L, L_0)$ be two simplicial maps. We say f and g are **contiguous as maps of pairs** if for every simplex $\sigma = v_0 \cdots v_p$ of K, the points $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$ span a simplex of L, and if σ is contained in K_0, then they span a simplex of L_0.

With maps that are contiguous as maps of pairs, we can extend the concept of chain homotopy to the case of relative homology, and how equal homomorphisms are induced on relative homology groups.

Theorem 12.6 [M] Let $f, g : (K, K_0) \to (L, L_0)$ be contiguous as maps of pairs. Then there exists a homomorphism $D : C_p(K, K_0) \to C_{p+1}(L, L_0)$ for all p such that

$$\partial D + D\partial = g_\# - f_\#.$$

Thus, f_* and g_* are equal as maps of the relative homology groups.

See [M] for proof details.

The main point is to notice that D maps $C_p(K_0)$ to $C_{p+1}(L_0)$.

Topological Invariance of Homology Groups

Want to show: $H_p(K)$ depends only on $|K|$, and not on the specific choice of K.

Method: We showed that a simplicial map $f : |K| \to |L|$ induces a homomorphism f_* of the homology groups. We want to argue that an arbitrary continuous map $h : |K| \to |L|$ can be approximated by a simplicial map f, and then argue that the induced homomorphism depends only on h, and not on the particular approximation chosen.

Simplicial Approximation

We present the concept of approximation in the context of simplicial complexes. Rather than specifying an error of approximation as is the practice in some other fields of mathematics, we present a condition defined using star of the vertices.

Def Let $h : |K| \to |L|$ be a continuous map. We say h satisfies the **star condition** relative to (or w.r.t.) K and L if for every vertex $v \in K^{(0)}$, there exists a vertex $w \in L^{(0)}$ such that

$$h(\mathrm{St}\, v) \subset \mathrm{St}\, w.$$

Lemma 14.1 [M] Let $h : |K| \to |L|$ satisfy the star condition relative to K and L. Choose $f : K^{(0)} \to L^{(0)}$ such that $\forall v \in K^{(0)}$, $h(\mathrm{St}\, v) \subset \mathrm{St}\, \underbrace{f(v)}_{w}$.

(a) For $\sigma \in K$, choose $\bar{x} \in \mathrm{Int}\,\sigma$ and $\tau \in L$ such that $h(\bar{x}) \in \mathrm{Int}\,\tau$. Then f maps each vertex of σ to a vertex of τ.

(b) f may be extended to a simplicial map of K into L, which we also call f.

(c) If $g : K \to L$ is another simplicial map such that $h(\mathrm{St}\, v) \subset \mathrm{St}\,(g(v))$ $\forall v \in K^{(0)}$, then f and g are contiguous.

Proof

(a) Let $\sigma = v_0 \cdots v_p$. Then $\bar{x} \in St\, v_i \ \forall i$. So

$$h(\bar{x}) \in h(St\, v_i) \subset St\, f(v_i) \quad \forall i.$$

So, $h(\bar{x})$ has positive barycentric coordinates w.r.t. each vertex $f(v_i)$, $i = 0, \dots, p$. These vertices must form a subset of the vertex set of τ.

(b) Straightforward.

(c) Since $h(\bar{x}) \in h(St\, v_i) \subset St(g(v_i)) \ \forall i$, the vertices $g(v_0), \dots, g(v_p)$ must also be vertices of τ. Thus $f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$ span a face of τ. □

We define the concept of simplicial approximation using the star condition.
More in the next lecture...