MATH 524: Lecture 15 (10/07/2025)

Today: * simplicial approximation
* subdivision

Recall $h: |K| \to |L|$ satisfies star condition: $h(\operatorname{St} v) \subset \operatorname{St} w$

Def Let $h: |K| \to |L|$ be a continuous map. If $f: K \to L$ is a simplicial map such that $h(\operatorname{St} v) \subset \operatorname{St} f(v)$ $\forall\, v \in K^{(0)}$, then f is called a **simplicial approximation** to h.

Intuitively, f is "close to" h in the following sense: given $\bar{x} \in |K|$, $\exists$ a simplex τ of L s.t. $h(\bar{x}), f(\bar{x}) \in \tau$. We formalize this concept now.

Lemma 14.2 [M] Let $f: K \to L$ be a simplicial approximation to $h: |K| \to |L|$. Given $\bar{x} \in |K|$, there exists a simplex $\tau \in L$ such that $h(\bar{x}) \in \operatorname{Int} \tau$, $f(\bar{x}) \in \tau$.

Proof Follows from Lemma 14.1 (a).

We can also compose simplicial approximations to get a simplicial approximation for the composition of continuous maps.

Theorem 14.3 [M] Let $h: |K| \to |L|$ and $k: |L| \to |M|$ have simplicial approximations $f: K \to L$ and $g: L \to M$, respectively. Then $g \circ f$ is a simplicial approximation to $k \circ h$.

Proof 1. $g \circ f$ is a simplicial map.

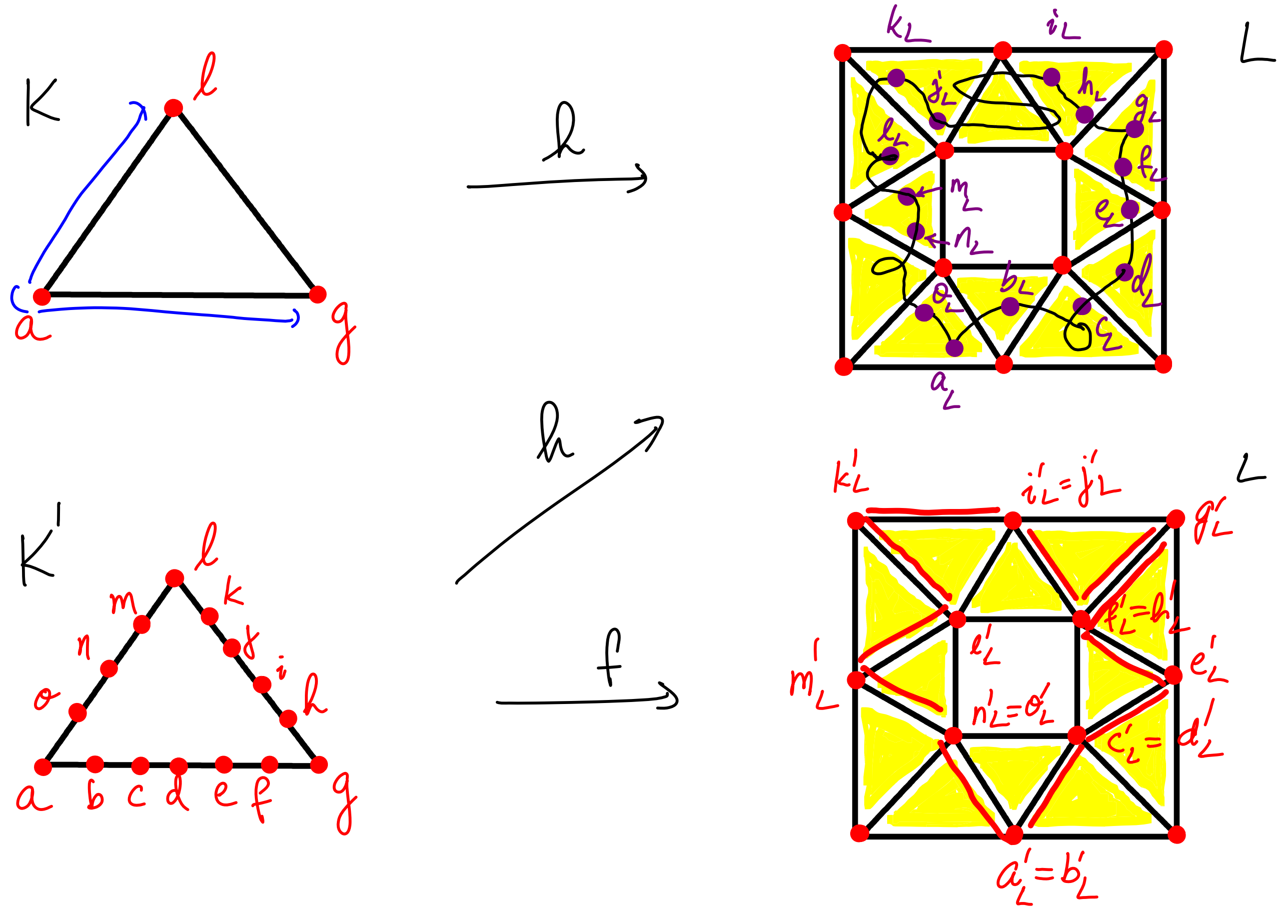
2. If $v \in K^{(0)}$, then $h(\operatorname{St} v) \subset \operatorname{St} f(v)$, as f is a simplicial approximation to h. Hence

$k(h(\operatorname{St} v)) \subset k(\operatorname{St} f(v)) \subset \operatorname{St}(g(f(v)))$,

as g is a simplicial approximation to k. $\square$

Example

$h(st(a,K)) \not\subset st(v,L)$ for any $v \in L^{(0)}$.



We consider K to be the 1-complex made of 3 1-simplices, and L to be the 2-complex that models an annulus. Let $h : |K| \to |L|$ map all of $|K|$ to the loop on $|L|$ as shown. We also consider a "refinement" of K by adding several more vertices to obtain K' such that $|K| = |K'|$. Hence, h applies without change to K'.

It is clear that h does not satisfy the star condition relative to K and L. Indeed, notice that $St(a,K) = K - \{\overline{lg}, l, g\}$, and there is no vertex in L such that $h(St(a,K))$ is a subset of its star in L.

But h does satisfy the star condition relative to K' and L. So h has a simplicial approximation $f: K' \to L$, and one such approximation is shown.

If $h: |K| \to |L|$ satisfies the star condition relative to K and L, there exists a well defined homomorphism

$$h_*: H_p(K) \to H_p(L) \quad \text{for all } p$$

obtained by setting $h_* = f_*$, where f is a simplicial approximation to h.

Not surprisingly, we can extend the star condition to the level of relative homology.

Lemma 14.4 [M] Let $h: |K| \to |L|$ satisfy the star condition relative to K & L, and suppose h maps $|K_0|$ into $|L_0|$.

(a) Any simplicial approximation $f: K \to L$ to h also maps $|K_0|$ into $|L_0|$. Also, the restriction of f to K_0 is a simplicial approximation to the restriction of h to $|K_0|$.

(b) Any two simplicial approximations f and g to h are contiguous as maps of pairs.

Subdivision

We had seen in the example that $h: |K| \to |L|$ did not satisfy the star condition relative to K and L, but it did relative to K' and L, where K' is a "finer" or "refined" version of K. We formalize this idea now, and talk about subdivisions.

We first formally define a subdivision. We then introduce barycentric subdivision as a "canonical" subdivision.

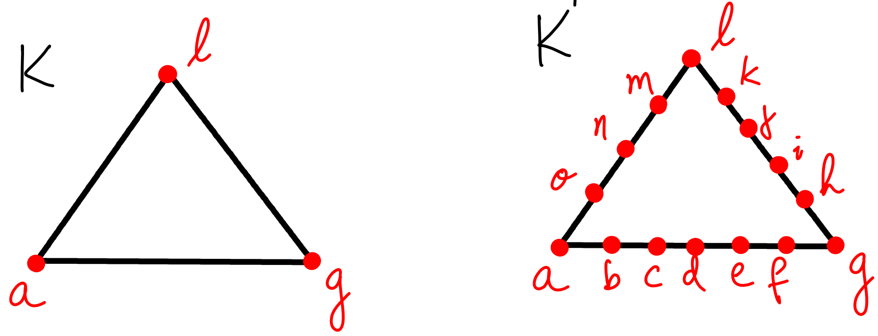
Def Let K be a geometric complex in $\mathbb{R}^d$. A complex K' is said to be a **subdivision** of K if

1. each simplex of K' is contained in a simplex of K, and
2. each simplex of K is the union of **finitely** many simplices of K'.

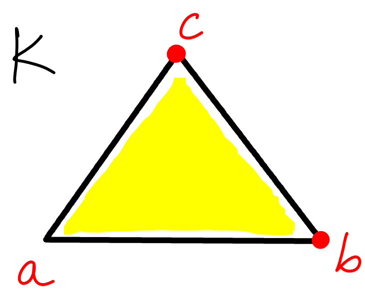
The conditions $\Rightarrow$ $|K|$ and $|K'|$ are equal as sets. The finiteness condition in 2. guarantees that $|K|$ and $|K'|$ are equal as topological spaces.

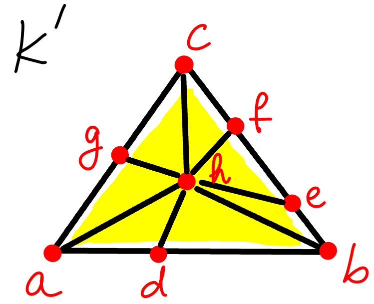
Examples

1.



2.





In 1 and 2 above, K' is a subdivision of K.

3. K : $[0,1]$ (1-simplex and its vertices)

K' : $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ $\forall n \in \mathbb{Z}_{>0}$, and their vertices, and the vertex 0.

$|K| = |K'|$ as sets, but they are not equal as topological spaces, as the finiteness requirement in condition 2 is violated. Hence K' is not a subdivision of K.

We get some results directly from the definition of subdivision.

<u>Properties</u>

1. If K'' is a subdivision of K', and K' is a subdivision of K, then K'' is a subdivision of K.

2. If K' is a subdivision of K, and $K_0 \subset K$ is a subcomplex, then the collection of simplices of K' that lie in $|K_0|$ is automatically a subdivision of K_0. We call this subdivision the subdivision of K_0 induced by K'.

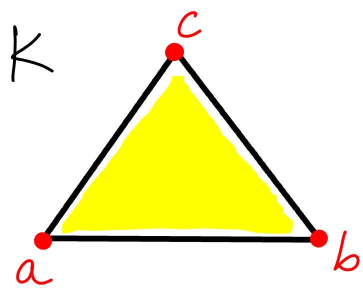
Subdivision satisfy a sort of "star condition", as the following lemma describes.

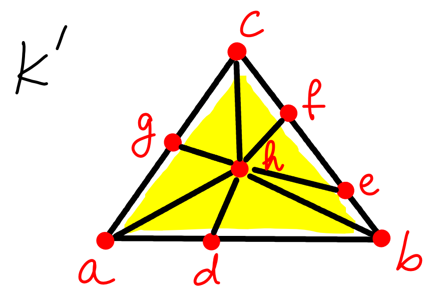
Lemma 15.1 [M] Let K' be a subdivision of K. Then for every $\bar{w} \in K'^{(0)}$, there exists a vertex $\bar{v} \in K^{(0)}$ such that

$$St(\bar{w}, K') \subset St(\bar{v}, K).$$

Indeed, if σ is a simplex in K s.t. $\bar{w} \in \operatorname{Int}\sigma$, then this inclusion holds precisely when $\bar{v}$ is a vertex of σ.

Example





Here, $St(h, K') \subset St(a, K)$, for instance.

Proof ($\Rightarrow$) (Straightforward). $\bar{w} \in St(\bar{w}, K')$ by definition. Hence by the given inclusion, $\bar{w}$ belongs to some open simplex of K, which has $\bar{v}$ as a vertex.

($\Leftarrow$) Let $\bar{w} \in \operatorname{Int}\sigma$, and $\bar{v}$ be a vertex of σ. Then we show that

$$|K| - St(\bar{v}, K) \subset |K| - St(\bar{w}, K')$$

Notice that $|K| - St(\bar{v}, K)$ is the union of all simplices in K that do not have $\bar{v}$ as a vertex. This is also a collection of simplices τ in K'. No such τ can have $\bar{w}$ as a vertex, as $\bar{w} \in \operatorname{Int}\sigma \subset St(\bar{v}, K)$. Hence any such τ lies in $|K| - St(\bar{w}, K')$. $\square$