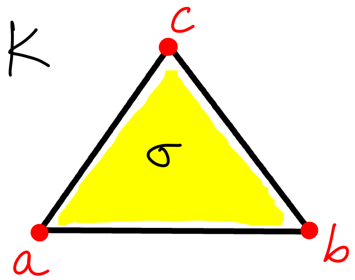
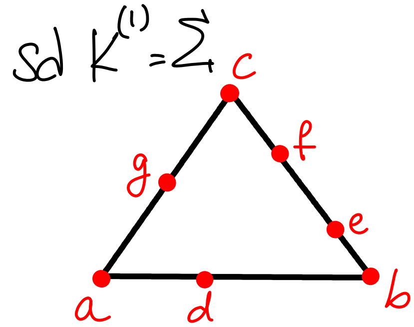
MATH 524: Lecture 16 (10/09/2025)

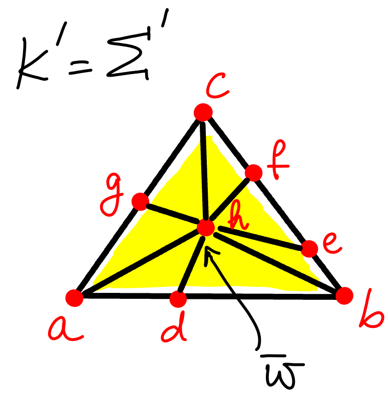
Today: * cone of K with vertex $\bar{w}$
* barycentric subdivision

We now consider ideas for how to construct subdivisions in general—one approach is to do it in increasing dimensions of the skeleton of the complex.

Back to Example 2:



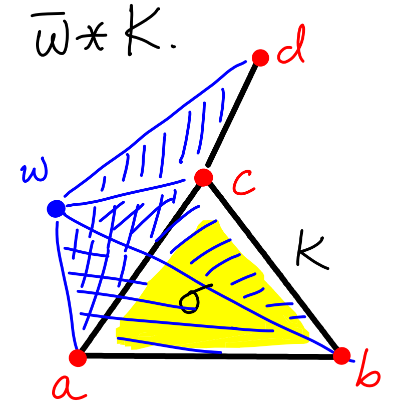




We can extend the subdivision Σ_1 of $K^{(1)}$ to that of $K^{(2)} = K$ by forming the **cone** $\bar{w} * \Sigma_1$, where $\bar{w}$ is any interior point of σ (here K is σ and its faces). In general, we can extend the subdivision Σ of $K^{(p)}$ to that of $K^{(p+1)}$ by forming the cone $\bar{w} * \Sigma$, where $\bar{w}$ is an interior point of the $(p+1)$-simplex σ.

Def Let K be a simplicial complex in $\mathbb{R}^d$, and $\bar{w} \in \mathbb{R}^d$ is a point such that each ray emanating from $\bar{w}$ intersects $|K|$ in at most one point. Then the **cone of K with vertex $\bar{w}$** is the collection of all simplices of the form $\bar{w}\bar{a}_0 \cdots \bar{a}_p$, where $\bar{a}_0 \cdots \bar{a}_p$ is a simplex of K, along with all faces of such simplices. We denote this collection as $\bar{w} * K$.

Example: Consider K to be the 2-complex shown—$\triangle abc$, edge $\overline{cd}$, and faces. Let $\bar{w}$ be a point lying "above" K. The cone $\bar{w} * K$ has the tetrahedron $wabc$, triangle wcd, and faces.



Note

1. $\bar{w} * K$ is indeed a well-defined simplicial complex, and has K as a subcomplex. We refer to K as the **base of the cone** $\bar{w} * K$.

2. $\dim(\bar{w} * K) = \dim(K) + 1$, as the ray intersection condition requires that $\bar{w} \notin \text{plane}(\sigma)\ \forall \sigma \in K$.

Back to example 2: Let the new subdivision of K be called Σ'. Then Σ' is obtained by "**starring** Σ from $\bar{w}$." → or "coning Σ from $\bar{w}$."

We can define the subdivision of K in an inductive fashion, going up one dimension at each step. We need a basic result first.

Lemma 15.2 [M] If K is a complex, the intersection of any collection of subcomplexes of K is a subcomplex of K. Conversely, if $\{K_\alpha\}$ is a collection of complexes in $\mathbb{R}^d$ and the intersection $|K_\alpha| \cap |K_\beta|$ is the polytope of a complex that is a subcomplex of both K_α and K_β for all α, β, then $\bigcup_\alpha K_\alpha$ is a complex.

We will use this lemma to justify how we define the subdivision in an inductive (or iterative) fashion. In particular, we star from one point within each simplex to the subdivision of its boundary.

Def Let K be a complex. Suppose L_p is the subdivision of $K^{(p)}$. Let σ be a $(p+1)$-simplex of K. $\mathrm{Bd}\,\sigma$ is a polytope of a subcomplex of $K^{(p)}$, and hence of L_p; we denote the latter by L_σ. For $\bar{w}_\sigma \in \mathrm{Int}\,\sigma$, the cone $\bar{w}_\sigma * L_\sigma$ is a complex whose underlying space is σ. We define L_{p+1} to be the union of L_p and the cones $\bar{w}_\sigma * L_\sigma$ as σ ranges over all $(p+1)$-simplices of K. L_{p+1} is the subdivision of $K^{(p+1)}$ obtained by **starring L_p from the points $\bar{w}_\sigma$.**

For the above definition to be correct, we need to verify that L_{p+1} is indeed a simplicial complex. To this end, we note the following facts.

(1) $|\bar{w}_\sigma * L_\sigma| \cap |L_p| = \mathrm{Bd}\,\sigma$ is the polytope of the subcomplex L_σ of both $\bar{w}_\sigma * L_\sigma$ and L_p.

(2) If τ is another $(p+1)$-simplex of K, then $|\bar{w}_\sigma * L_\sigma|$ and $|\bar{w}_\tau * L_\tau|$ intersect in the simplex $\sigma \cap \tau$ of K, which is the polytope of a subcomplex of L_p, and hence of both L_σ and L_τ. Hence it follows from Lemma 15.2 that L_{p+1} is a simplicial complex.

How do we choose the point $\bar{w}_\sigma$ for each σ? While there are (infinitely) many choices, we can use a "canonical" choice.

Def The **barycenter** of $\sigma = v_0 \dots v_p$ is defined to be the point

$$\hat{\sigma} = \sum_{i=0}^{p} \frac{1}{(p+1)} v_i .$$

$\hat{\sigma}$ is the point of $\mathrm{Int}\,\sigma$ all of whose barycentric coordinates with respect to the vertices of σ are equal.

σ: 0-simplex $\Rightarrow$ $\hat{\sigma} = \sigma$

1-simplex $\Rightarrow$ $\hat{\sigma}$: midpoint

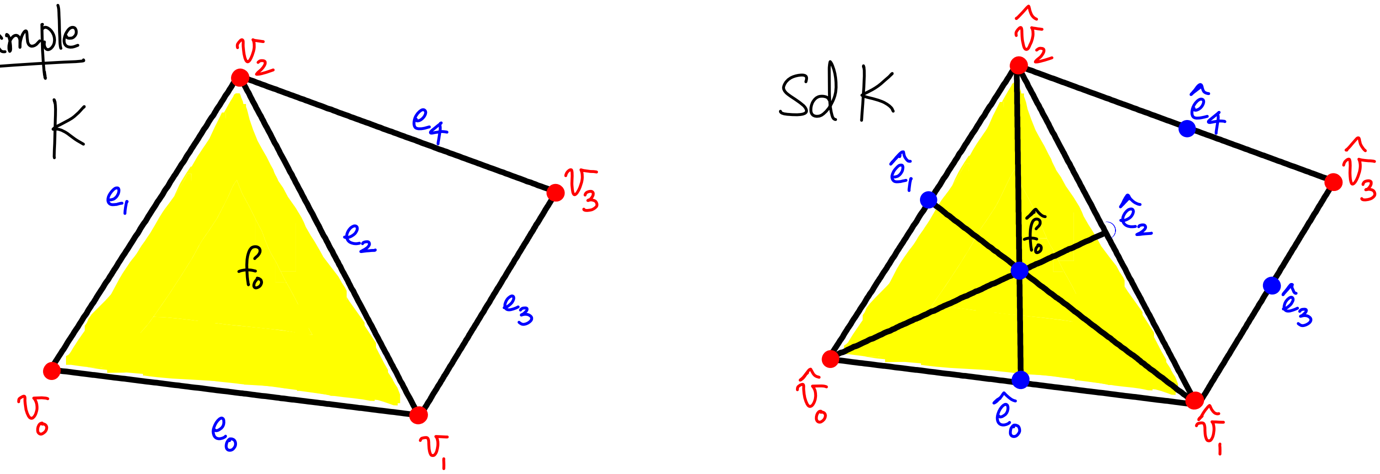
p-simplex $\Rightarrow$ $\hat{\sigma}$: centroid of σ. $p \geq 1$

We star from the barycenters to construct the barycentric subdivision.

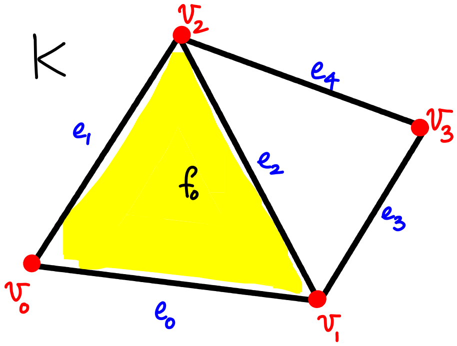
Def Let K be a simplicial complex. Let $L_0 = K^{(0)}$. In general, L_p is the subdivision of the p-skeleton of K. Let L_{p+1} be the subdivision of $K^{(p+1)}$ obtained by starring L_p from the barycenters of all (p+1)-simplices of K. By Lemma 15.2, the union of the complexes L_p is a subdivision of K. This is the **first barycentric subdivision** of K, denoted **Sd K**.

The first barycentric subdivision of $\mathrm{Sd}\,K$, denoted $\mathrm{Sd}(\mathrm{Sd}\,K)$ or Sd^2K, is the second barycentric subdivision of K. Similarly, we define Sd^rK, the r^{th} barycentric subdivision for any integer $r \geq 0$, with $\mathrm{Sd}^0K = K$.

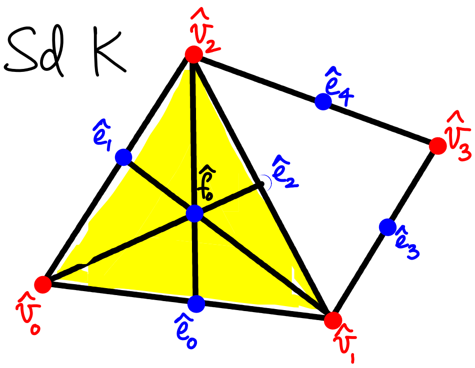
Example



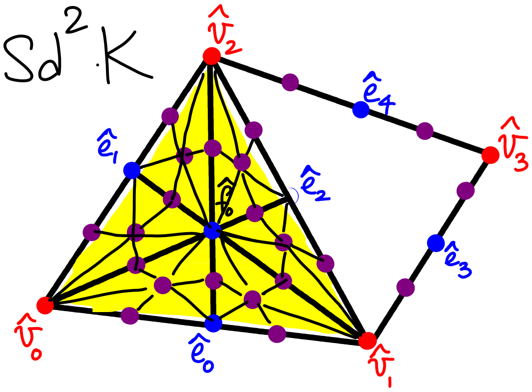
K



Sd K



$Sd^2 K$



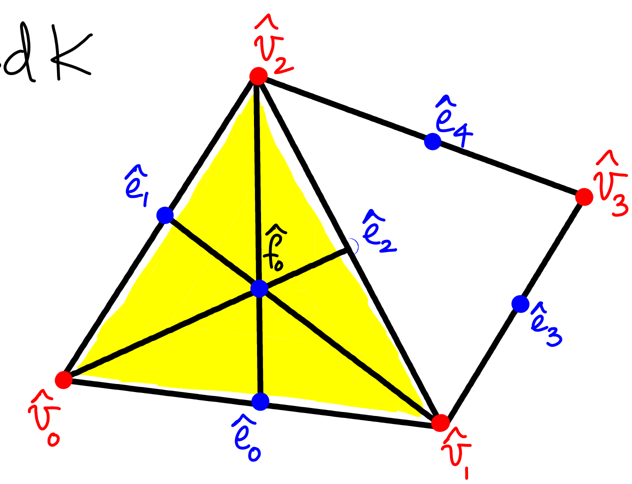
Explicit Description of the simplices in Sd K

Notation $\sigma_1 > \sigma_2$ means σ_2 is a proper face of σ_1, or equivalently, σ_1 is a proper coface of σ_2.

Lemma 15.3 [M] Sd K is the collection of simplices of the form $\hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_p$ where $\sigma_1 > \sigma_2 > \cdots > \sigma_p$.

Illustration

Sd K



The edges in Sd K are of the form $\hat{e}_j \hat{v}_i$ where $e_j > v_i$; or of the form $\hat{f}_0 \hat{e}_j$ where $f_0 > e_j$. Similarly, the triangles in Sd K are of the form $\hat{f}_0 \hat{e}_j \hat{v}_i$ where $f_0 > e_j > v_i$.

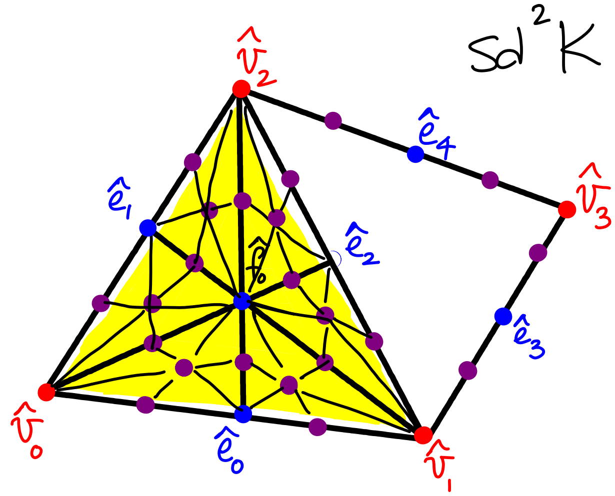
<u>Proof</u> (by induction)

True for $K^{(0)}$ (as $\hat{v} = v \ \forall v \in K^{(0)}$).

Now suppose each simplex of Sd K lying in $|K^{(p)}|$ is of this form. Let τ be a simplex of Sd K lying in $|K^{(p+1)}|$, but not in $|K^{(p)}|$. Then τ belongs to one of the complexes $\hat{\sigma} * L_\sigma$, where σ is a $(p+1)$-simplex of K, and L_σ is the first barycentric subdivision of the complex made of the proper faces of σ. By induction, each simplex of L_σ is of the form $\hat{\sigma}_0 \hat{\sigma}_1 \cdots \hat{\sigma}_p$. Then τ must be of the form $\hat{\sigma} \hat{\sigma}_0 \cdots \hat{\sigma}_p$. □

Notice that the simplices in $Sd^2 K$ are much "smaller" than the simplices in Sd K. This observation is formalized in the following theorem.

$Sd^2 K$



Theorem 15.4 [M] Given a finite complex K, a metric for $|K|$, and $\varepsilon > 0$, there exists an r such that each simplex in $Sd^r K$ has diameter less than ε.

Def For a subset S of a metric space (X, d), its **diameter** is $\mathrm{diam}(S) = \sup\{d(\bar{x}, \bar{y}) \mid \bar{x}, \bar{y} \in S\}$.

↳ the metric of the metric space.

See [M] for the proof.