MATH 524: Lecture 17 (10/14/2025)

Today: * Simplicial approximation
* exact sequences

Simplicial Approximation

We now talk about how to use subdivision to find a simplicial approximation of any continuous function $h: |K| \to |L|$.

Recall: A simplicial approximation of a continuous map $h: |K| \to |L|$ by a simplicial map $f: K \to L$ satisfies

$$h(\operatorname{St} v) \subset \operatorname{St}(f(v)) \ \forall\, v \in K^{(0)}.$$

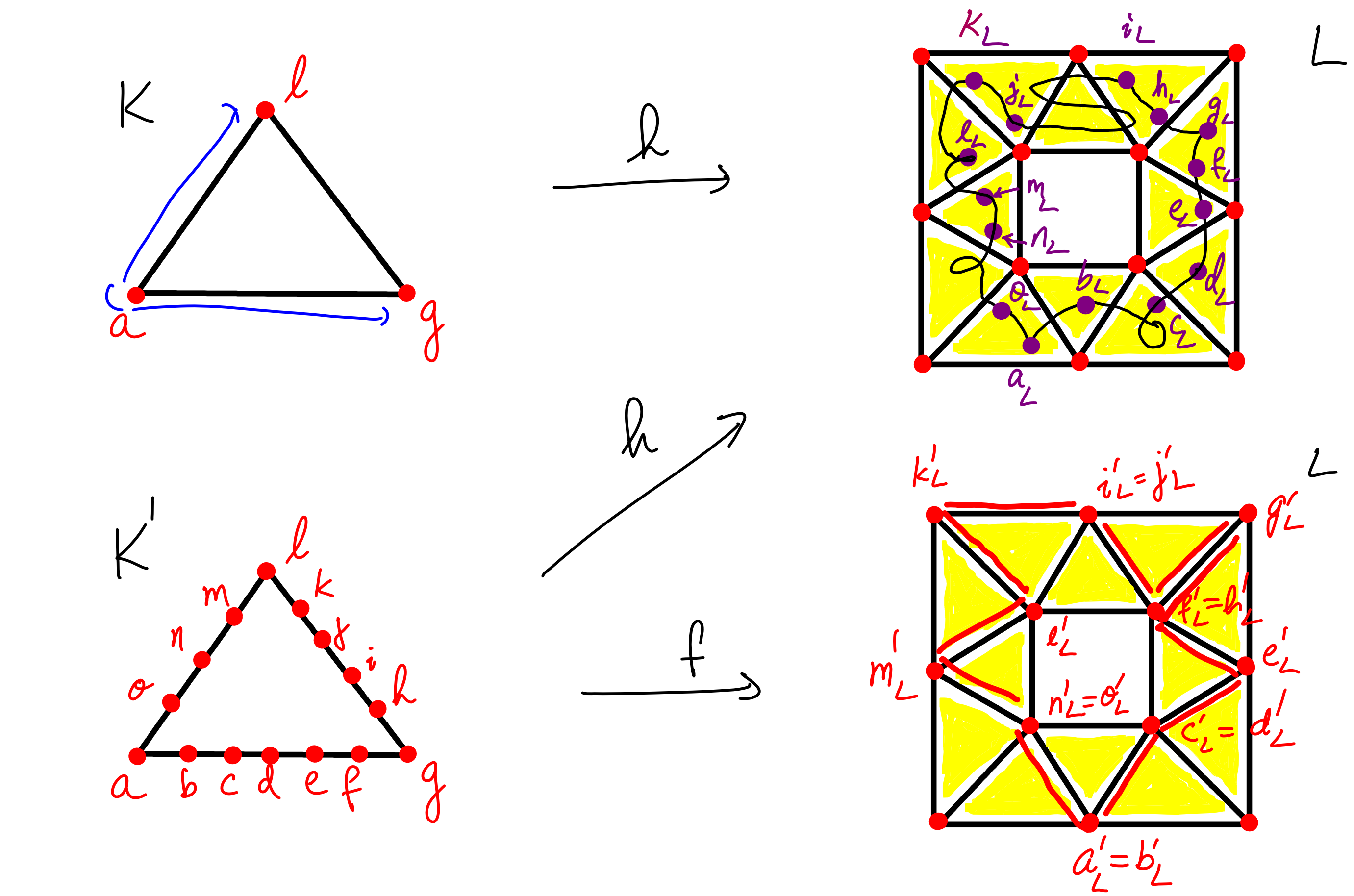
We had also seen that homomorphisms $f_\#$ associated with simplicial maps $f: K \to L$ induce isomorphisms at the homology level. Our ultimate goal is to argue that the homology groups are determined by the underlying spaces, rather than specific choices of the complex.

We now look at the next step toward that goal – to show that we can always approximate a continuous map by a simplicial map once we have a fine enough subdivision of the original complex.

The result for the case when K is finite is quite accessible compared to that when K is infinite. We will discuss only the finite case in detail here.

Theorem 16.1 [M] **(The finite simplicial approximation theorem)**
Let K and L be complexes, and let K be finite. Given a continuous map $h: |K| \to |L|$, there is an r such that h has a simplicial approximation $f: \operatorname{Sd}^r K \to L$.

Here is an illustration we already saw in Lecture 15.



Of course, K' is not a barycentric subdivision of K here. But the example illustrates the result nonetheless. The key idea is that by subdividing K enough, we could approximate h by a simplicial map from K' to L.

Even in the illustration shown, one could argue that f misses the detail in h in some places, e.g., between i_L and j_L where h looks like "S", while f looks like "•". But one could consider a finer subdivision K'' of K' where we have some more vertices between i and j. The image under f could be closer to h in that case.

In this context, the barycentric subdivision is just one kind of subdivision we could use. At the same time, its nice structure makes it convenient to devise proofs of results. On the other hand, $Sd^r K$ might produce "bad" simplices, e.g., triangles that are too skinny. There are other classes of subdivision where the triangles produced are "round" (while still holding a small diameter).

<u>Proof</u> Cover $|K|$ by open sets $h^{-1}(\mathrm{St}\,\overline{w})$, as $\overline{w}$ ranges over $L^{(0)}$. Let this covering be called $\mathcal{A}$. Then $\mathcal{A}$ is an open covering of the compact metric space $|K|$. So, there exists a number λ such that any set of $|K|$ with diameter less than λ lies in one of the elements of $\mathcal{A}$. This number is called a **Lebesgue number** for $\mathcal{A}$.

Here is the standard argument for why a Lebesgue number should exist in this case.

Suppose there does not exist a Lebesgue number for $\mathcal{A}$. Then we can choose a sequence C_n of sets such that $\operatorname{diam}(C_n) < \frac{1}{n}$, but C_n does not lie in any element of $\mathcal{A}$. Choose $\overline{x}_n \in C_n$. By compactness, some subsequence $\{\overline{x}_{n_i}\}$ converges, to say, $\overline{x}$. Now, $\overline{x} \in A$ for some $A \in \mathcal{A}$. As A is open, it contains C_{n_i} for i sufficiently large — a contradiction.

Back to the main proof now...

Choose r s.t. each simplex σ in $Sd^r K$ has $\mathrm{diam}(\sigma) < \lambda/2$. Then each $St\,\bar{v}$ for $\bar{v} \in (Sd^r K)^{(0)}$ has diameter $< \lambda$. So, it ($St\,\bar{v}$) lies in one of the sets $h^{-1}(St\,\bar{w})$. So, $h: |K| \to |L|$ satisfies the star condition relative to $Sd^r K$ and L, and hence a simplicial approximation exists. $\square$

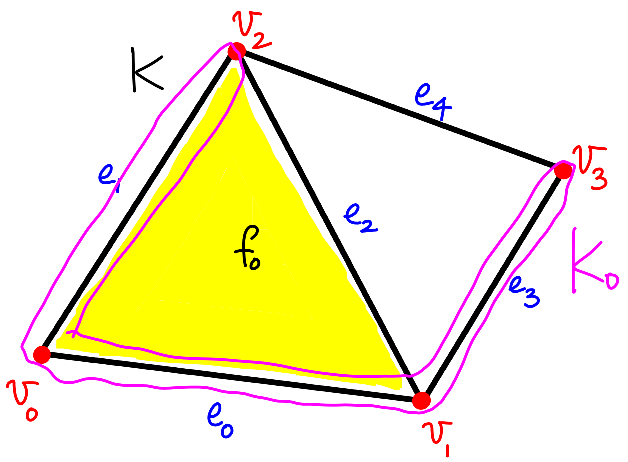
Extending the simplicial approximation theorem to the case when K is not finite ($h: |K| \to |L|$) is much more involved.

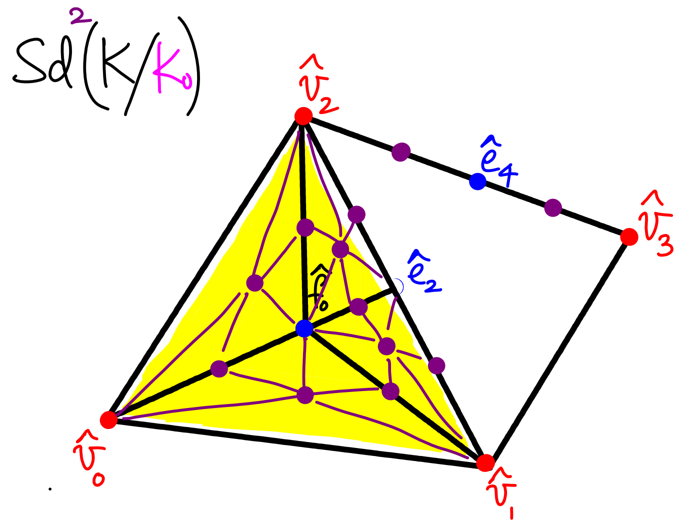
We introduce a key technique related to subdivision used in this process. In particular, the default barycentric subdivision will not work.

Subdividing K while keeping K_0 (a subcomplex) fixed

Def Here is a sequence of subdivisions of skeletons of K. Let $J_0 = K^{(0)}$. In general, J_p is a subdivision of $K^{(p)}$, and each simplex τ of K_0 with $\dim(\tau) \leq p$ belongs to J_p. Define J_{p+1} to be the union of J_p, all $\tau \in K_0$ with $\dim(\tau) = p+1$, and the cones $\hat{\sigma} * J_\sigma$ as σ ranges over all $(p+1)$-simplices of K **not in** K_0. Here J_σ is a subcomplex of J_p whose polytope is $Bd\,\sigma$. The union of all complexes $J_p\ \forall p$ is a subdivision of K, denoted $Sd\,(K/K_0)$, and is called the **first barycentric subdivision of K, holding K_0 fixed.**

We define $Sd^r(K/K_0)$ similarly: $Sd^2(K/K_0) = Sd\left(Sd(K/K_0)/K_0\right)$, for instance.





K_0: {edges e_0, e_1, e_3, and all v_j}.

$Sd(K/K_0)$ and $Sd^2(K/K_0)$

We finish by listing the main result. See [M] for proof.

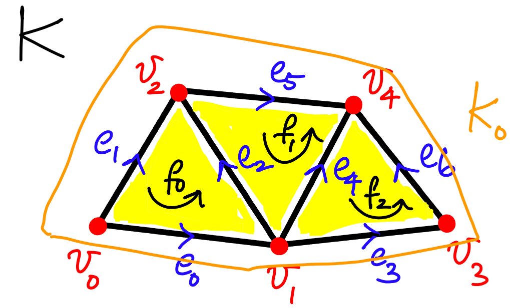
Theorem 16.5 [M] (**The general simplicial approximation theorem**)

Let K and L be complexes, and let $h: |K| \to |L|$ be a continuous map. Then there exists a subdivision K' of K such that h has a simplicial approximation $f: K' \to L$.

That's all we will cover in this subtopic. Next we move on to an important algebraic technique — exact sequences.

Exact Sequences

What are the relationships between $H_p(K,K_0)$, $H_p(K)$, and $H_p(K_0)$?



Example (same as example 3 in lecture 9)

Here, $H_2(K,K_0) \simeq \mathbb{Z}$. $\bar{r} = \sum_{i=0}^{2} \bar{f}_i$ is a generator.

Also, $H_1(K_0) \simeq \mathbb{Z}$, $\{\bar{z}\}$ is a basis, where $\bar{z} = \bar{e}_0 + \bar{e}_3 + \bar{e}_6 - \bar{e}_5 - \bar{e}_1$.

So $H_2(K,K_0) \simeq H_1(K_0)$.

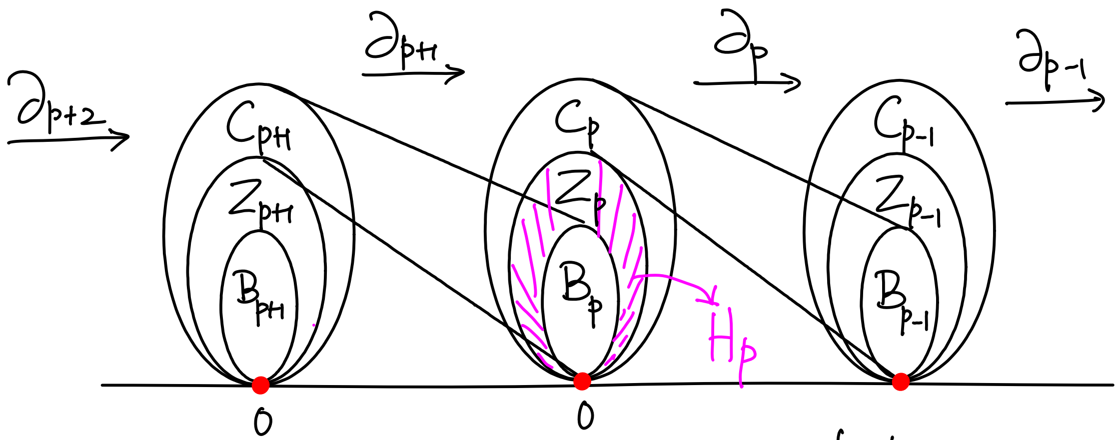
It turns out that $H_2(K,K_0) \simeq H_1(K_0)$ here is not a mere coincidence. To present the general result, we first need to introduce the algebraic machinery of exact sequences – of objects (think groups, rings, etc.) and maps (homomorphisms) between them.

Def Consider a sequence (finite or infinite) of groups and homomorphisms

$$\cdots \xrightarrow{\phi_{i-2}} A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \rightarrow \cdots$$

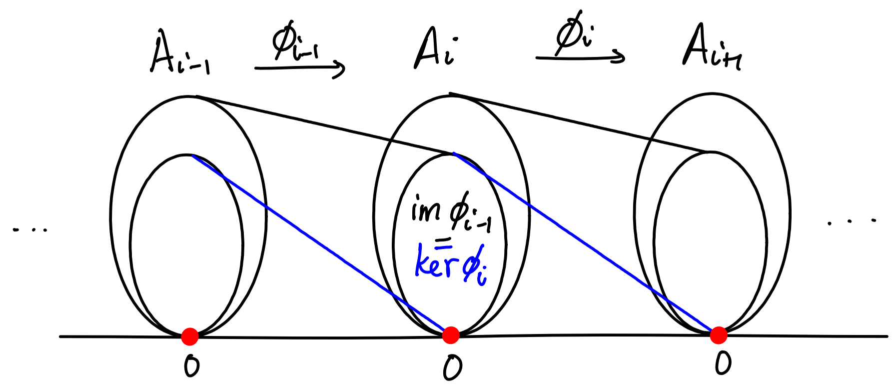
This sequence is **exact** at A_i if image ϕ_{i-1} = kernel ϕ_i. If it is exact everywhere, it is an **exact sequence**. Exactness is not defined at the first and last group of the sequence, if they exist.

The sequence we have seen already, of chain groups and boundary homomorphisms, is **not** exact!



The indices are decreasing left to right here, but that is not an issue. Indeed, notice that $\operatorname{im} \partial_{p+1} = B_p \neq \ker \partial_p = Z_p$.

Here is the picture of exact sequences that we want.



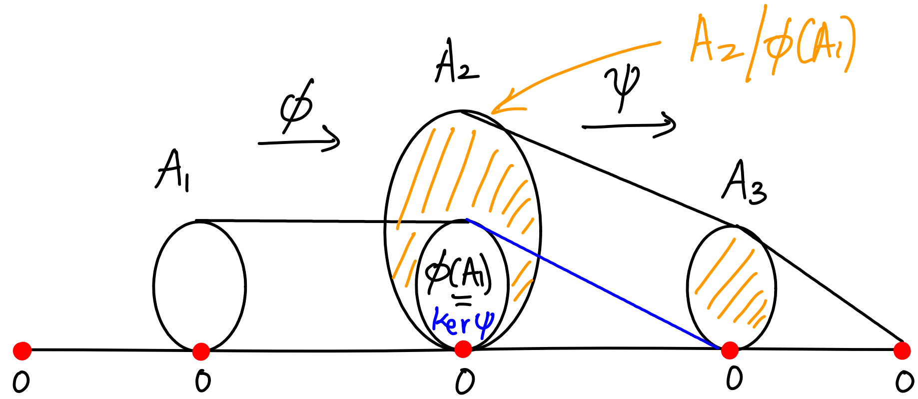
Several results on exact sequences (with abelian groups)

$0 \to$ denotes the trivial group

1. $A_1 \xrightarrow{\phi} A_2 \longrightarrow 0$ is exact iff ϕ is an epimorphism (surjective/onto).

2. $0 \longrightarrow A_1 \xrightarrow{\phi} A_2$ is exact iff ϕ is a monomorphism (injective/1-to-1).

3. Suppose the sequence $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ is exact. Such a sequence is called a **short exact sequence** (SES).



Then $A_2/\phi(A_1) = \operatorname{cok}\psi$ is isomorphic to A_3; this isomorphism is induced by ψ. Conversely, if $\psi: A \to B$ is an epimorphism with $\ker\psi = K$, then the sequence

$$0 \to K \xrightarrow{i} A \xrightarrow{\psi} B \to 0$$

is exact, where i is inclusion.

4. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\phi} A_3 \xrightarrow{\beta} A_4$ is exact. Then the following statements are equivalent.
   (i) α is an epimorphism.
   (ii) β is a monomorphism.
   (iii) ϕ is the zero homomorphism.

5. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \to A_3 \to A_4 \xrightarrow{\beta} A_5$ is exact. Then so is the induced sequence $0 \to \operatorname{cok}\alpha \to A_3 \to \ker\beta \to 0$.

It will be instructive to draw diagrams similar to the base case for each of these results!

Def Consider two sequences of groups and homomorphisms having the same index set.

$$\begin{tikzcd} \cdots \arrow[r] & A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] & A_{i+1} \arrow[r, "\phi_{i+1}"] \arrow[d, "\alpha_{i+1}"] & \cdots \\ \cdots \arrow[r] & B_i \arrow[r, "\psi_i"] & B_{i+1} \arrow[r, "\psi_{i+1}"] & \cdots \end{tikzcd}$$

A **homomorphism of the first sequence into the second** is a family of homomorphisms $\alpha_i : A_i \to B_i$ such that each square of maps

$$\begin{tikzcd} A_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] \arrow[dr, phantom, "\square"] & A_{i+1} \arrow[d, "\alpha_{i+1}"] \\ B_i \arrow[r, "\psi_i"] & B_{i+1} \end{tikzcd}$$

commutes, i.e., $\alpha_{i+1} \circ \phi_i = \psi_i \circ \alpha_i$.

It is an isomorphism of sequences if each α_i is an isomorphism.