MATH 524: Lecture 18 (10/16/2025)

Today: * chain complex and chain map
* connecting homomorphism

Recall: homomorphism of one sequence into a second..

We had studied simplicial maps (from K to L), and associated homomorphisms between the chain groups (and its subgroups) in both complexes. Indeed, that set up illustrates the above definition. At the same time, it turns out we could study such collections of groups and homomorphisms in a much more general setting — and not necessarily on a simplicial complex.

Example: chain maps between chain complexes, which we define now.

Def A **chain complex** $\mathcal{C}.$ is a family $\{C_p, \partial_p\}$ of abelian groups C_p and homomorphisms ∂_p
$\partial_p : C_p \to C_{p-1}$ such that $\partial_p \circ \partial_{p+1} = 0 \ \forall p$.

The group $H_p(\mathcal{C}) = \ker \partial_p / \operatorname{im} \partial_{p+1}$ is the p-th **homology group** of the chain complex $\mathcal{C}$.

Notice that the chain, cycle, boundary, and homology groups, along with boundary homomorphisms does indeed fit this framework — and hence the overloading of notation! At the same time, chain complexes could be much more general! We do need $\partial_p \circ \partial_{p+1} = 0 \ \forall p$ in the general setting.

Now consider two chain complexes $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C_p', \partial_p'\}$. We can define a family of homomorphisms from C_p to C_p' with additional requirements on "connecting" them to ∂_p and ∂_p' as follows. We define $\phi_p : C_p \to C_p'$ to be the homomorphism from the p^{th} abelian group of $\mathcal{C}$ to the p^{th} abelian group of $\mathcal{C}'$, for each p.

ϕ_p should be such that each "square" in the diagram commutes.

ϕ satisfies $\partial_p' \circ \phi_p = \phi_{p-1} \circ \partial_p \ \forall p$.

$$\begin{array}{lccccccc}
\mathcal{C} & \longrightarrow & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} & \longrightarrow \cdots \\
& & \downarrow \phi_{p+1} & & \phi_p \downarrow & \boxed{} & \downarrow \phi_{p-1} & \\
\mathcal{C}' & \longrightarrow & C_{p+1}' & \xrightarrow{\partial_{p+1}'} & C_p' & \xrightarrow{\partial_p'} & C_{p-1}' & \longrightarrow \cdots
\end{array}$$

$\partial_p' \circ \phi_p = \phi_{p-1} \circ \partial_p$

Then entire family of homomorphisms, $\phi_p \ \forall p$, is referred to as a **chain map** from $\mathcal{C}$ to $\mathcal{C}'$ — $\phi : \mathcal{C} \to \mathcal{C}'$.

Recall how we talked about simplicial maps inducing homomorphisms at the homology level. We get the same result in the general setting as well.

A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ induces a homomorphism $(\phi_*)_p : H_p(\mathcal{C}) \longrightarrow H_p(\mathcal{C}')$.

We now introduce one more concept related to short exact sequences.

Def Consider a short exact sequence

$$0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0.$$

This sequence is said to **split** if the group $\phi(A_1)$ is a direct summand in A_2. So the sequence becomes

$$0 \to A_1 \xrightarrow{\phi} \phi(A_1) \oplus B \xrightarrow{\psi} A_3 \to 0.$$

where ϕ defines an isomorphism of A_1 with $\phi(A_1)$, and ψ defines an isomorphism of B with A_3.

We end by stating two results on short exact sequences that split. See [M] for details and proofs.

Theorem 23.1 [M] Let $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ be exact. Then the following statements are equivalent.

(1) The sequence splits.

(2) There exists a map $p : A_2 \to A_1$ such that $p \circ \phi = i_{A_1}$
identity in A_1

(3) There exists a map $j : A_3 \to A_2$ such that $\psi \circ j = i_{A_3}$.
identity in A_3

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, shift left, "\phi"] & A_2 \arrow[l, shift left, "p"] \arrow[r, shift left, "\psi"] & A_3 \arrow[l, shift left, "j"] \arrow[r] & 0 \end{tikzcd}$$

Corollary 23.2 [M]. Let $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ be exact. If A_3 is free abelian, the sequence splits.

Exact homology sequence of a pair K, K_0

Goal: Connect $H_p(K,K_0)$, $H_p(K)$, $H_p(K_0)$

We first need to define a homomorphism connecting $H_p(K,K_0)$ and $H_{p-1}(K_0)$

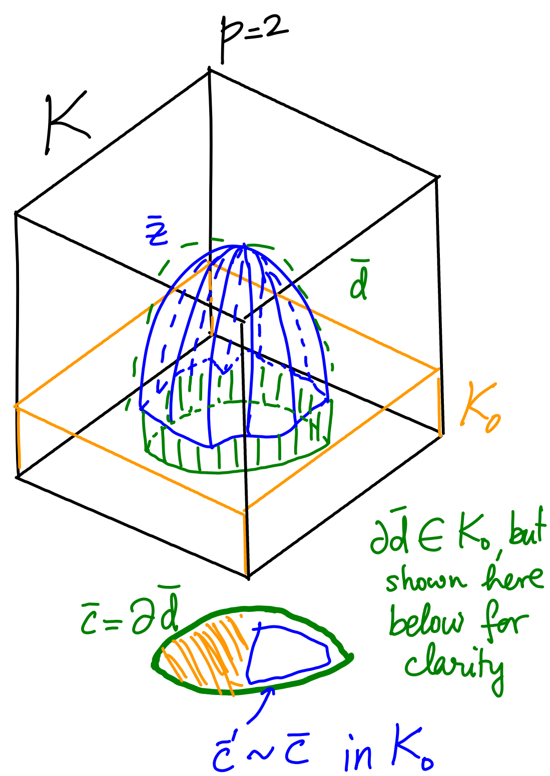
$$\partial_* : H_p(K,K_0) \longrightarrow H_{p-1}(K_0)$$

We call this homomorphism the **homology boundary homomorphism** or the **connecting homomorphism**.

Consider a cycle $\bar{z} \in C_p(K,K_0)$.

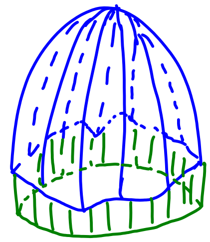
We consider the class $\{\bar{z}\}$ as the coset modulo $C_p(K_0)$ of a p-chain $\bar{d}$ of K such that $\partial \bar{d}$ is carried by K_0. Notice that $\partial \bar{d}$ is automatically a $(p-1)$-cycle of K_0. We define

$$\partial_* \{\bar{z}\} = \{\partial \bar{d}\}$$

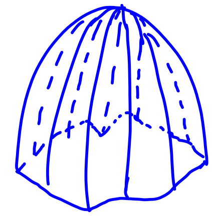


We detail the algebraic construction/definition in this fashion.

$\bar{d}$

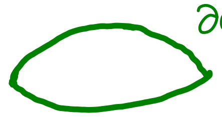


$\bar{z}$



$$\begin{tikzcd} & C_p(K) \arrow[r, "\pi_\#"] \arrow[d, "\partial"] & C_p(K,K_0) \\ C_{p-1}(K_0) \arrow[r, "i_\#"] & C_{p-1}(K) & \end{tikzcd}$$

$\pi_\#(\bar{d}) = \bar{z}$



$\partial\bar{d} \in K_0$

$\bar{c}$
$(p-1)$-cycle

$\{\bar{c}\} =: \partial_*\{\bar{z}\}$

$i : K_0 \to K$ and $\pi : (K,\phi) \to (K,K_0)$ are inclusions.

$i_\#$ is an inclusion, $\pi_\#$ is projection of $C_p(K)$ onto $C_p(K)/C_p(K_0)$.

So we define $\partial_*\{\bar{z}\}$ by a "zig-zag" process.

Def A **long exact sequence** is an exact sequence whose index set is $\mathbb{Z}$. So the sequence is infinite in both directions. It could begin or end with an infinite string of trivial groups.

Theorem 23.3 [M] **(The exact homology sequence of a pair)**

Let K_0 is a subcomplex of K. There is a long exact sequence

$$\cdots \to H_p(K_0) \xrightarrow{i_*} H_p(K) \xrightarrow{\pi_*} H_p(K,K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

where $i: K_0 \to K$ and $\pi: (K,\phi) \to (K,K_0)$ are inclusions and ∂_* is the connecting homomorphism. There exists a similar long exact sequence in reduced homology.

$$\cdots \to \tilde{H}_p(K_0) \xrightarrow{i_*} \tilde{H}_p(K) \xrightarrow{\pi_*} H_p(K,K_0) \xrightarrow{\partial_*} \tilde{H}_{p-1}(K_0) \longrightarrow \cdots$$

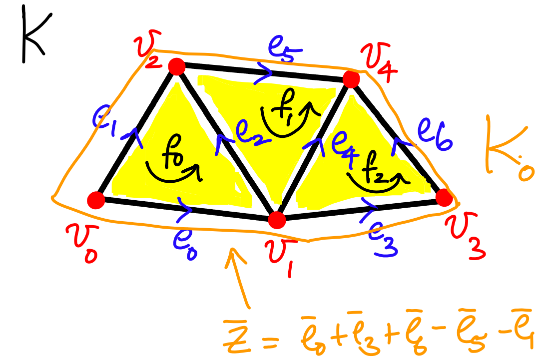
It turns out $\tilde{H}_p(K,K_0) = H_p(K,K_0)$ as long as $K_0 \neq \phi$. Essentially, relative homology groups are already reduced.

One direct use of the above result is in figuring out the structure of $H_p(K,K_0)$ when the structures of $H_p(K)$ and $H_p(K_0)$ are known. In many cases, the latter homology groups could be characterized more easily, and hence could be used in conjunction with this exact homology sequence to identify $H_p(K,K_0)$.

We apply this result to a few examples.

1. We had seen that (in Lecture 12)

$H_2(K,K_0) \simeq \mathbb{Z}$ with $\bar{r} = \sum_{i=0}^{2} f_i$ being a generator.



Also, $H_1(K_0) \simeq \mathbb{Z}$ with $\bar{z}$ being a generator.

Notice that $\partial \bar{r} = \bar{z}$. In this case $\partial_*: H_2(K,K_0) \to H_1(K_0)$ is an isomorphism. We could reach the same conclusion using the exact sequence result. A portion of the long exact sequence is

$$\underbrace{H_2(K)}_{=0} \longrightarrow H_2(K,K_0) \xrightarrow{\partial_*} H_1(K_0) \longrightarrow \underbrace{H_1(K)}_{=0}.$$

$H_2(K)$ and $H_1(K)$ are both trivial, and hence ∂_* is both a monomorphism and an epimorphism, i.e., it's an isomorphism.

There are no 2-cycles to start with.

Notice that any 1-cycle in K is also a 1-boundary. More intuitively, K has no holes.

Recall results 1 and 2 from Lecture 18 on exact sequences!