# MATH 524: Lecture 19 (10/21/2025)

Today: * exact sequences of chain complexes
* zigzag lemma, diagram chasing

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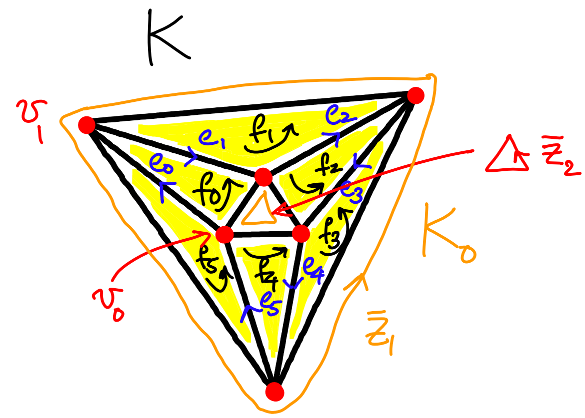
**Recall:** Exact homology sequence of a pair $K, K_0$.

$$\cdots \to H_p(K_0) \xrightarrow{i_*} H_p(K) \xrightarrow{\pi_*} H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

2. Consider the annulus we saw in Lecture 12.




$H_2(K, K_0) = ?$ $\quad H_1(K, K_0) = ?$

Consider reduced homology (for $\tilde{H}_0(K_0)$).

$K_0$ consists of the outer and inner perimeters, both oriented CCW.

Recall that with $\bar{\gamma} = \sum_{i=0}^{5} \bar{f}_i$, $\partial \bar{\gamma} = \bar{z}_1 - \bar{z}_2$.

Also, $\partial_1 \bar{e}_0 = v_1 - v_0$.

We consider the relevant portion of the exact homology sequence:

$$H_2(K) \xrightarrow{0} H_2(K, K_0) \xrightarrow{(\partial_*)_2} H_1(K_0) \xrightarrow{(i_*)_1} H_1(K) \xrightarrow{(\pi_*)_1} H_1(K, K_0) \xrightarrow{(\partial_*)_1} \tilde{H}_0(K_0) \xrightarrow{(i_*)_0} \tilde{H}_0(K)$$

$$0 \longrightarrow \underset{\mathbb{Z}}{?} \longrightarrow \underset{\{\bar{z}_1\}, \{\bar{z}_2\}}{\mathbb{Z} \oplus \mathbb{Z}} \longrightarrow \underset{\{\bar{z}_1\} \text{ or } \{\bar{z}_2\}}{\mathbb{Z}} \longrightarrow \underset{\mathbb{Z}}{?} \longrightarrow \underset{\{v_1 - v_0\}}{\mathbb{Z}} \longrightarrow 0$$

19.2

If $i : K_0 \to K$ is inclusion, $i_*$ maps both $\{\bar{z}_1\}$ and $\{\bar{z}_2\}$ to, say, $\{\bar{z}_1\}$. So $(i_*)_1$ is an epimorphism, and $\ker (i_*)_1 \simeq \mathbb{Z}$, and it is generated by $\{\bar{z}_1\} - \{\bar{z}_2\}$. Hence, we get that $(\pi_*)_1$ is the zero homomorphism. Equivalently, notice that any $\bar{z} \in H_1(K)$ is homologous to $\bar{z}_1$ (or $\bar{z}_2$), so is projected out by $\pi_*$ in $H_1(K, K_0)$.

So, we have

$$\xrightarrow{0} H_1(K, K_0) \xrightarrow{(\partial_*)_1} \underset{\mathbb{Z}}{\tilde{H}_0(K_0)} \xrightarrow{0} 0$$

$\Rightarrow (\partial_*)_1$ is an isomorphism, so $H_1(K, K_0) \simeq \mathbb{Z}$. It is generated by, e.g., $\{\bar{e}_0\}$ with $\partial \bar{e}_0 = v_1 - v_0$.

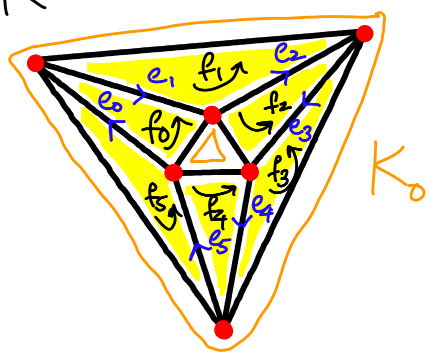
Again, by applying results 1 and 2 from Lecture 19 on exact sequences here, we notice $(\partial_*)_1$ is both an epimorphism and a monomorphism.

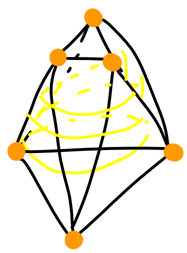
We also get $\operatorname{im}(\partial_*)_2 = \ker (i_*)_1$ and $(\partial_*)_2 : H_2(K, K_0) \to \ker (i_*)_1$ is an isomorphism. Hence $H_2(K, K_0) \simeq \mathbb{Z}$. It is generated by $\bar{r} = \sum_{i=0}^{5} \bar{f}_i$, as $\partial_2 \bar{r} = \bar{z}_1 - \bar{z}_2$, which in turn generates $\ker (i_*)_1$, as we noted previously.

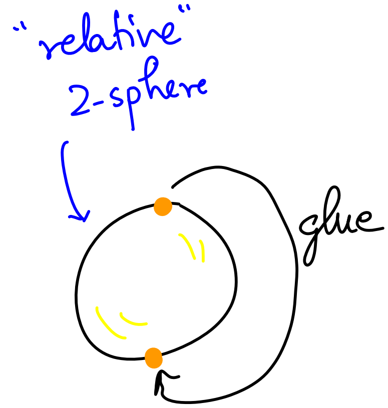
While the ~~for~~ formal method works, it also helps to think intuitively, how the complex looks after "shrinking" all of $K_0$ to a point.

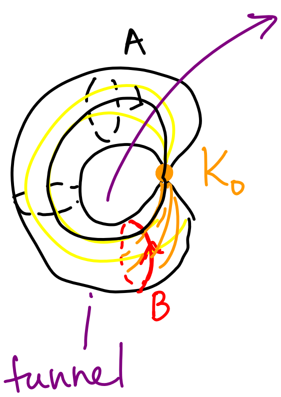
Think about shrinking both $\bar{z}_1$ and $\bar{z}_2$ (which comprise $K_0$) to a point each, and then "gluing" these two points.

K

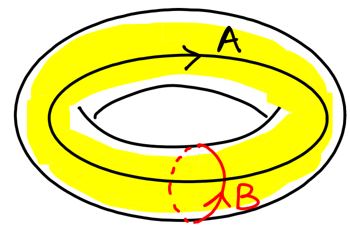





← This object is the pinched torus. Compare this object to the torus.




Notice that while the tunnel loop (A) still exists, the handle loop (B) is now a boundary - it bounds the two chain from the pinched point (representing $K_0$) to B (looks like a cap). Hence, $H_1(K, K_0) \simeq \mathbb{Z}$.

Also, there is still one enclosed space, or void, and hence $H_2(K, K_0) \simeq \mathbb{Z}$ as well here.

**Recall:** chain complexes and chain maps

We had introduced the (far more) general concept of chain complexes and chain maps between them. A chain complex $\mathcal{C}$ consists of a set of objects (groups, for instance) with maps (homomorphisms) between them that satisfy the condition that composition of consecutive maps is trivial (i.e, zero).

We have $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C_p', \partial_p'\}$, with $\partial_p' \circ \partial_{p+1}' = 0$. A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ is a family of homomorphisms that commutes with $\partial_p$, $\partial_p'$, i.e.,

$$\partial_p' \circ \phi_p = \phi_{p-1} \circ \partial_p \quad \forall\, p.$$

Each "square" commutes:

$$\begin{tikzcd} \cdot \arrow[r, "\partial_p"] \arrow[d, "\phi_p"'] & \cdot \arrow[d, "\phi_{p-1}"] \\ \cdot \arrow[r, "\partial_p"] & \cdot \end{tikzcd}$$

So, cycle (boundaries) in $\mathcal{C}$ get mapped to cycles (boundaries) in $\mathcal{C}'$, and $\phi$ induces a homomorphism of the homology groups

$$(\phi_*)_p : H_p(\mathcal{C}) \longrightarrow H_p(\mathcal{C}').$$

Notice that we can define $Z_p' = \ker \partial_p'$, $B_p' = \operatorname{im} \partial_{p+1}'$, and $H_p' = Z_p'/B_p'$ for $\mathcal{C}'$.

We present the result on existence of long exact sequences given a family of short exact sequences in the general setting of chain complexes.

**Notation** $\mathcal{C}, \mathcal{D}, \mathcal{E}$ : chain complexes

$$\mathcal{C} = \{C_p, \partial_C\}, \quad \mathcal{D} = \{D_p, \partial_D\}, \quad \mathcal{E} = \{E_p, \partial_E\}$$

groups in the chain complexes

homomorphisms for each chain complex

We will supress listings of subscripts to avoid clutter.

**Def** Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be chain complexes and $0$ denote the trivial chain complex whose groups vanish in each dimension. Let $\phi: \mathcal{C} \to \mathcal{D}$ and $\psi: \mathcal{D} \to \mathcal{E}$ be chain maps. We say the sequence $\mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E}$ is **exact** at $\mathcal{D}$ if $\ker \psi_p = \operatorname{im} \phi_p \ \forall p$, i.e., if the sequence $C_p \xrightarrow{\phi_p} D_p \xrightarrow{\psi_p} E_p$ is exact $\forall p$.

We say the sequence $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$ is a **short exact sequence of chain complexes** if in each dimension $p$, the sequence

$$0 \to C_p \xrightarrow{\phi} D_p \xrightarrow{\psi} E_p \to 0$$

is an exact sequence of groups.

**Example** Let $K_0 \subseteq K$ be a subcomplex of simplicial complex $K$. Then the sequence

$$0 \longrightarrow \mathcal{C}(K_0) \xrightarrow{i} \mathcal{C}(K) \xrightarrow{\pi} \mathcal{C}(K, K_0) \longrightarrow 0$$

is exact, as $C_p(K, K_0) = C_p(K)/C_p(K_0)$ by definition.

We have $\ker \pi_p = \operatorname{im} i_p \ \forall p$.

Here $\mathcal{C}(K) = \{C_p(K), \partial_p\}$, $\mathcal{C}(K_0) = \{C_p(K_0), \partial_p\}$, and so on. Notice that we directly get the following results:

$\mathcal{C}(K_0) \longrightarrow \mathcal{C}(K)$ is injective and
$\mathcal{C}(K) \longrightarrow \mathcal{C}(K, K_0)$ is surjective.

We can construct/define connecting homomorphisms using which we can connect such short exact sequences of chain complexes to build long exact sequences of chain complexes. Recall the result from the previous lecture about long exact sequences for homology groups of a pair $(K, K_0)$ — we will see that this result follows as a direct instance of the more general result specified on chain complexes and chain maps. We first state the general result, and come back to the above example to illustrate the same.

**Lemma 24.1 [M]** (The **zig-zag lemma**) or (Snake lemma).
Suppose one is given chain complexes $\mathcal{C} = \{C_p, \partial_C\}$, $\mathcal{D} = \{D_p, \partial_D\}$, and $\mathcal{E} = \{E_p, \partial_E\}$, and chain maps $\phi, \psi$ such that the sequence $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$ is exact. Then there is a **long exact homology sequence**

$$\cdots H_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \xrightarrow{\phi_*} H_{p-1}(\mathcal{D}) \to \cdots$$

where $\partial_*$ is the **connecting homomorphism** and is induced by the boundary operator in $\mathcal{D}$ $(\partial_D)$.

<u>Back to the example</u> on long exact sequence of homology.
We just saw that the sequence

$$0 \longrightarrow \mathcal{C}(K_0) \xrightarrow{i} \mathcal{C}(K) \xrightarrow{\pi} \mathcal{C}(K, K_0) \longrightarrow 0$$

is exact. The exactness in the middle follows from the fact that a chain of $K$ is carried by $K_0$ iff it is zero in $C(K, K_0)$.

So Lemma 24.1 implies the existence of a long exact homology sequence of pair $(K, K_0)$:

$$\cdots \to H_p(K_0) \longrightarrow H_p(K) \longrightarrow H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \longrightarrow \cdots$$

Proof (sketch).

Main step: define connecting homomorphism $\partial_*$. We illustrate the technique of "diagram chasing" here — it's applied in more general settings (and not just to simplicial complexes).

$$\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] & D_{p+1} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\boxed{\phantom{x}}_2"] & E_{p+1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_p \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\boxed{\phantom{x}}_3"] & D_p \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\boxed{\phantom{x}}_0"] & E_p \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\boxed{\phantom{x}}_1"] & D_{p-1} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] & E_{p-1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "\phi"] & D_{p-2} \arrow[r, "\psi"] & E_{p-2} \arrow[r] & 0
\end{tikzcd}$$

$d_p$ $e_p$ $c_{p-1}$ $\partial_D d_p$ $\partial_C c_{p-1}$

$\psi \partial_D = \partial_E \psi$

$\boxed{\phantom{x}}_i$ : squares are indexed in the order in which they're used in the proof.

<u>Step 1</u>: (defining $\partial_*$). Given a cycle $e_p$ in $E_p$, since $\psi$ is surjective, we can choose $d_p \in D_p$ such that $\psi(d_p) = e_p$. Since $\boxed{\phantom{x}}_0$ commutes, the element $\partial_D d_p$ of $D_p$ lies in $\ker \psi$, as

$$\psi(\partial_D d_p) = \partial_E(\psi(d_p)) = \partial_E(\underbrace{e_p}_{\text{cycle}}) = 0.$$

Therefore, there exists $c_{p-1} \in C_{p-1}$ such that $\phi(c_{p-1}) = \partial_D d_p$ as $\ker \psi = \operatorname{im} \phi$. Since $\phi$ is injective, $c_{p-1}$ is unique here.
Further, $c_{p-1}$ is a cycle here, since

$$\phi(\partial_C c_{p-1}) = \partial_D \phi(c_{p-1}) = \partial_D(\partial_D d_p) = 0,$$

as $\square_1$ commutes. Again, since $\phi$ is injective, $\partial_C c_{p-1} = 0$.

We define $\partial_* \{e_p\} = \{c_{p-1}\}$, where $\{\cdot\}$ means "homology class of."

We'll present the rest of the proof in the next lecture...