MATH 524: Lecture 1 (08/19/2025)

This is Algebraic Topology.
I'm Bala Krishnamoorthy (Call me Bala).

Today:
* syllabus, logistics
* neighborhoods, continuous functions
* topology using neighborhoods
* homeomorphism

I will be teaching computational topology (Math 529) next semester. The two classes – Math 524 and Math 529 will be kept independent. In particular, we will spend nearly no focus on computational aspects in Math 524.

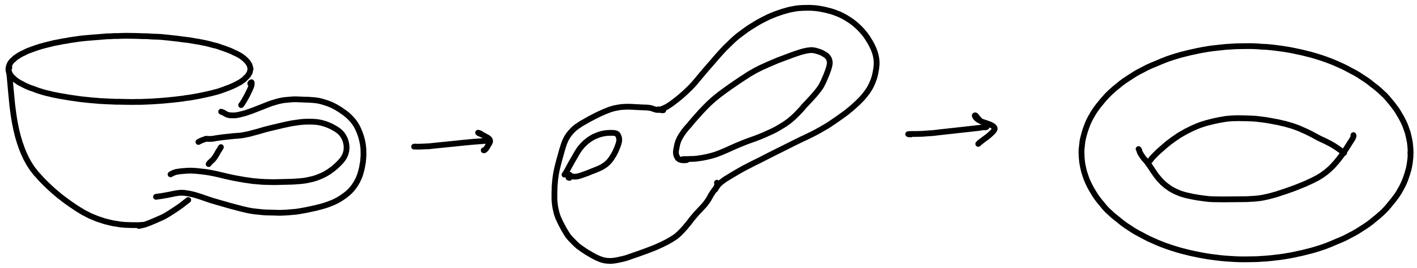
Check the course web page at

https://bala-krishnamoorthy.github.io/Math524.html

All documents, important updates, homework assignments etc. will be posted there. Check the class page frequently.

More about the video assignment to come soon. But you're encouraged to start looking for topics that you might want to make the video on as we proceed in the course.

Topology studies how "space is connected". You might have heard the (true!) joke that the topologist cannot distinguish between a coffee cup and a donut! Indeed, they both are connected the same way.



In algebraic topology, we cast problems on how space is connected as equivalent problems on algebraic objects – groups, rings, etc., and maps between them (homomorphisms).

As a subfield of mathematics, algebraic topology started in late 19th and early 20th century. Poincaré introduced the fundamental group first. Later Betti introduced homology groups, which are much easier to compute (both by hand as well as algorithmically) than the former.

We will spend a lot of time talking about homology groups, and the dual concept of cohomology. We will not be spending much attention on the fundamental group. There are several (equivalent) ways to define homology groups. Perhaps the "nicest" way to do so is using simplicial complexes. We will spend a fair bit of time studying simplicial homology.

We will introduce/refresh background concepts as needed. First, we will talk about continuous functions and topological spaces, defined in terms of neighborhoods.

Continuous functions

We first give the classical ε-δ definition in Euclean spaces.

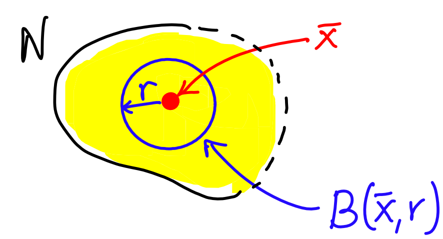
Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$. f is continuous at $\bar{x} \in \mathbb{R}^n$ if there exists $\delta > 0$ for every $\varepsilon > 0$ such that $\|f(\bar{y}) - f(\bar{x})\| < \varepsilon$ whenever $\|\bar{y} - \bar{x}\| < \delta$ for $\bar{y} \in \mathbb{R}^n$. f is continuous (in all of $\mathbb{R}^n$) if it is so at every $\bar{x} \in \mathbb{R}^n$.

my notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\mu}$, etc., are all vectors — lower case letters with a bar.

We give an equivalent definition based on neighborhoods.

Def A subset N of $\mathbb{R}^n$ is a **neighborhood** of $\bar{x} \in \mathbb{R}^n$ if for some $r > 0$, the closed ball $B(\bar{x}, r)$ centered at $\bar{x}$ is contained entirely within N.

Notice that neighborhood N can be open or closed.



$$B(\bar{x}, r) = \{\bar{y} \in \mathbb{R}^n \mid \|\bar{x} - \bar{y}\| \leq r\}$$

closed Ball of radius r centered at $\bar{x}$

Def $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous if given any $\bar{x} \in \mathbb{R}^n$ and a neighborhood N of $f(\bar{x})$ in $\mathbb{R}^m$, $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in $\mathbb{R}^n$.

Now we define what a topological space (or topology) is. We give the definition in terms of neighborhoods first. In most textbooks, you will see the definition given in terms of open sets. Later, we will see that both definitions are equivalent.

more notation: Upper case letters, e.g., A, B, X, Y, etc, denote sets or matrices.

Topological space (or topology)

Def I We are given a set X and a nonempty collection of subsets of X for each $\bar{x} \in X$ called the neighborhoods of $\bar{x}$. This is a topological space if it satisfies the following axioms.

(a) $\bar{x}$ lies in each of its neighborhood.

(b) Intersection of two neighborhoods of $\bar{x}$ is itself a neighborhood of $\bar{x}$.

(c) If N is a neighborhood of $\bar{x}$, and $\mathcal{U} \subseteq X$ contains N, then $\mathcal{U}$ is a neighborhood of $\bar{x}$.

(d) If N is a neighborhood of $\bar{x}$, $\mathring{N}$, the **interior** of N is also a neighborhood of $\bar{x}$.

The interior of N is $\mathring{N} = \{\bar{y} \in N \mid N \text{ is a neighborhood of } \bar{y}\}$.

Intuitively, every point of N not on its boundary is in its interior.

We can extend the definition of continuous functions to functions defined between topological spaces.

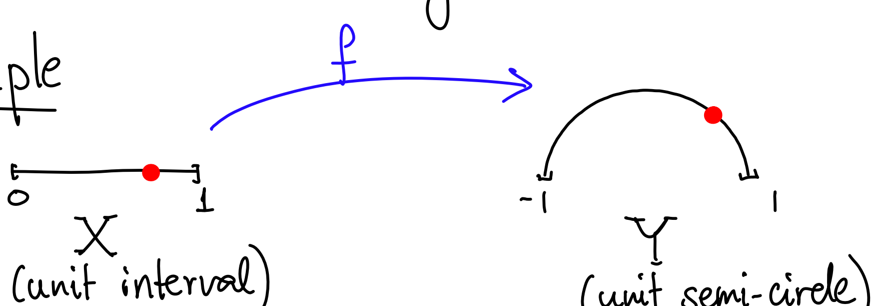
Def Let X, Y be topological spaces. $f: X \to Y$ is continuous if $\forall\, \bar{x} \in X$ and for every neighborhood N of $f(\bar{x})$ in Y, the set $f^{-1}(N)$ is a neighborhood of $\bar{x}$ in X.

We are interested in studying when two topological spaces are similar. There are a few different notions of topological similarity, and the strongest notion is that of homeomorphism. For two spaces to be homeomorphic, we need a function between them that is "nicer" than just a continuous function.

Def A function $f: X \to Y$ is a **homeomorphism** if it is one-to-one, onto, continuous, and has a continuous inverse.

When such a function exists between two spaces X and Y, we say they are **homeomorphic**, or are topologically equivalent. We denote this fact by $X \approx Y$.

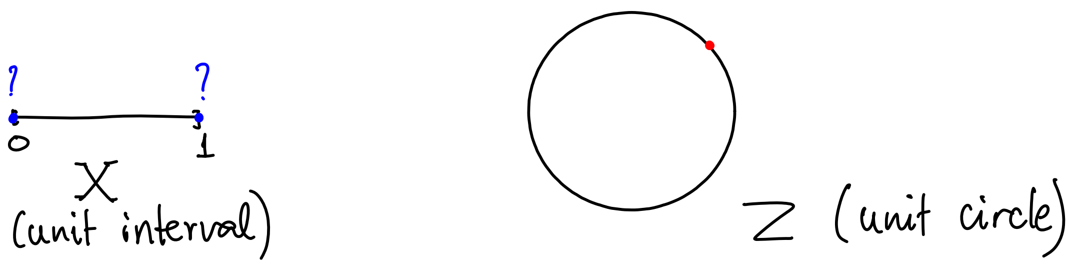
Example



$X \approx Y$. Can you define the function f?

Think of X & Y as subsets of $\mathbb{R}^2$, and write down the form of f, as well as f^{-1}. You can show f satisfies all requirements for being a homeomorphism.

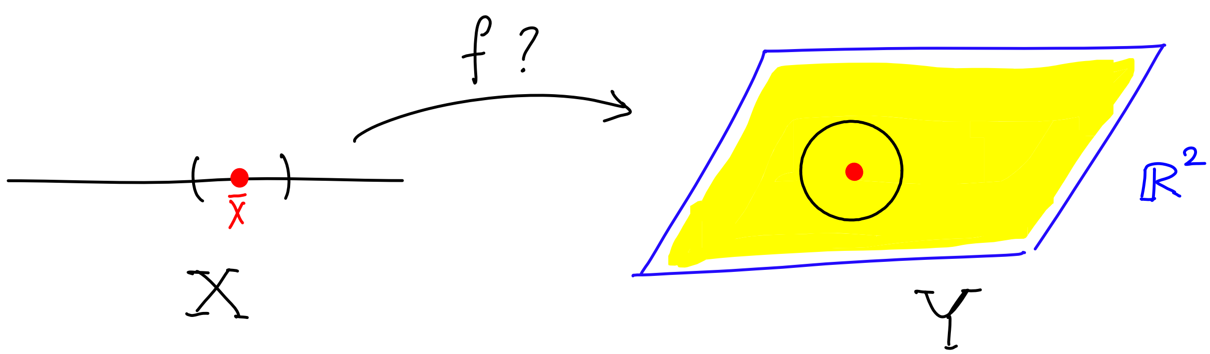
Showing two spaces are **not** homeomorphic could be harder — we need to show that no such function exists between X and Y.



Here, $X \not\cong Z$. Where do things breakdown?

Intuitively, one can notice the two end points of X behave distinctly from any point in Y.

Here is another example. Perhaps the simplest example of a topological space is $\mathbb{R}^d$ under the usual definition of neighborhoods, which specifies that any set $N \subseteq \mathbb{R}^d$ big enough to contain $B(\bar{x}, r)$ for some $r > 0$ is a neighborhood of $\bar{x} \in \mathbb{R}^d$. But notice that $\mathbb{R}^1 \not\cong \mathbb{R}^2$, for instance. It is not straightforward to prove this fact rigorously. But, how would one "argue" for it?



One method is to appeal to how the two spaces are connected. Recall that topologically similar spaces are "connected" the same way. Here, if we remove one point from both $X = \mathbb{R}^1$ and $Y = \mathbb{R}^2$, we can see that it affects the connectivity differently. Removing one point leaves X disconnected (into two pieces). But removing a point from Y still leaves it connected – its just like poking a hole in the "sheet" that is $\mathbb{R}^2$, which remains connected.

More formally, we could try to define a homeomorphism from X to Y. But we can observe that neighborhoods in X are 1-dimensional, while those in Y are 2D. Hence we cannot define a bijection between them.

We will talk about open sets in the next lecture, and define a topology using open sets. That definition is equivalent to the one introduced earlier today, i.e., Def I.