MATH 524: Lecture 20 (10/23/2025)

Today: $*$ more on zig-zag lemma
$*$ "stacking" sequences of chain complexes

Recall: proof of zigzag lemma... Step 1: Define $\partial_*\{e_p\} = \{c_{p-1}\}$.

Step 2 Show ∂_* is well defined — independent of the choice of $e_p \in \ker \partial_E$ and choice of c_{p-1} from $\{c_{p-1}\}$.

Recall that we defined ∂_* on homology classes — $\partial_*\{e_p\} = \{c_{p-1}\}$ for cycle $e_p \in E_p$ and corresponding cycle $c_{p-1} \in C_{p-1}$. We want to now show that this definition is independent of the choice of e_p and c_{p-1}. To this end, we start with cycles e_p, e_p' in E_p $(e_p, e_p' \in \ker \partial_E : E_p \rightarrow E_{p-1})$. We assume that $e_p \sim e_p'$ (homologous), and then argue that $c_{p-1} \sim c_{p-1}'$.

Given $e_p \sim e_p'$, we can find $e_{p+1} \in E_{p+1}$ such that $e_p - e_p' = \partial_E e_{p+1}$ (by definition of homology). Using the upper portion of the diagram, we argue that we can find $c_p \in C_p$ such that $c_{p-1} - c_{p-1}' = \partial_C c_p$.

$$\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] & \overset{d_{p+1}}{D_{p+1}} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\square_2"] & \overset{e_{p+1}}{E_{p+1}} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & \overset{c_p}{C_p} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\square_3"] & \overset{d_p, d_p'}{D_p} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] \arrow[dr, phantom, "\square_0"] & \overset{e_p \sim e_p'}{E_p} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r, "c_{p-1}' \sim"] & \overset{c_{p-1}}{C_{p-1}} \arrow[r, "\phi"] \arrow[d, "\partial_C"'] \arrow[dr, phantom, "\square_1"] & \overset{\partial_D d_p}{D_{p-1}} \arrow[r, "\psi"] \arrow[d, "\partial_D"'] & E_{p-1} \arrow[r] \arrow[d, "\partial_E"'] & 0 \\
0 \arrow[r] & C_{p-2} \arrow[r, "\phi"] & D_{p-2} \arrow[r, "\psi"] & E_{p-2} \arrow[r] & 0
\end{tikzcd}$$

$\square_i$: squares are indexed in the order in which they're used in the proof.

ψ is surjective. So choose d_p, d_p' such that $\psi(d_p) = e_p$ and $\psi(d_p') = e_p'$. Using the same arguments in Step 1, choose c_{p-1} and c_{p-1}' such that $\phi(c_{p-1}) = \partial_D d_p$ and $\phi(c_{p-1}') = \partial_D d_p'$.

Suppose $e_p - e_p' = \partial_E e_{p+1}$. $\overbrace{\text{Choose } d_{p+1} \in D_{p+1}}^{\text{recall that } \psi \text{ is surjective}}$ such that $\psi(d_{p+1}) = e_{p+1}$. Notice that

$$\psi(d_p - d_p' - \partial_D d_{p+1}) = e_p - e_p' - \overbrace{\partial_E \psi}^{\text{as } \square_2 \text{ commutes, } \psi \partial_D = \partial_E \psi}(d_{p+1})$$

$$= e_p - e_p' - \partial_E e_{p+1} = 0.$$

So $d_p - d_p' - \partial_D d_{p+1} \in \ker \psi : D_p \to E_p$. By exactness, it should also be in $\operatorname{im} \phi : C_p \to D_p$.

So we can choose $c_p \in C_p$ such that $\phi(c_p) = d_p - d_p' - \partial_D d_{p+1}$.

So $\phi(\partial_C c_p) = \partial_D \phi(c_p)$ as $\boxed{}_3$ commutes, $\phi \partial_C = \partial_D \phi$

$$= \partial_D(d_p - d_p' - \partial_D d_{p+1}) = \phi(c_{p-1} - c_{p-1}').$$

But ϕ is injective, so $\partial_C c_p = c_{p-1} - c_{p-1}'$. So $c_{p-1} \sim c_{p-1}'$.

We need to show also that ∂_* is indeed a homomorphism. Notice that $\psi(d_p + d_{p'}) = e_p + e_{p'}$ and $\phi(c_{p-1} + c_{p-1}') = \partial_D(d_p + d_{p'})$. So $\partial_*\{e_p + e_p\} = \{c_{p-1} + c_{p-1}'\}$ by definition, and the latter part equals $\partial_*\{e_p\} + \partial_*\{e_p'\}$.

Thus, $\partial_*\{e_p + e_p'\} = \partial_*\{e_p\} + \partial_*\{e_p'\}$, showing ∂_* is a homomorphism.

Steps 3, 4, 5 Prove exactness at $H_p(\mathcal{D})$, $H_p(\mathcal{E})$, and $H_{p-1}(\mathcal{C})$.

See [M] for details. □

Notice how we zig-zag down and to the left to go from e_p to c_{p-1} in the process of defining $\partial_*\{e_p\}$. Hence the name "zig-zag" or "snake" lemma.

It turns out we can extend this type of results on existence of long exact sequences with connecting homomorphisms to pairs (or more) of exact sequences of chain complexes.

Theorem 24.2 [M] Suppose we are given a commutative diagram

internal zig-zag (in "top floor")

$$\begin{tikzcd}
 & c_{p-1} & d_p & e_p & \\
0 \arrow[r] & \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\boxed{}_2"] & \mathcal{D} \arrow[r, "\psi"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\boxed{}_1"] & \mathcal{E} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \mathcal{C}' \arrow[r, "\phi'"] & \mathcal{D}' \arrow[r, "\psi'"] & \mathcal{E}' \arrow[r] & 0 \\
 & c'_{p-1} & d'_p & e'_p &
\end{tikzcd}$$

internal zig-zag (in "bottom floor")

where horizontal sequences are exact sequences of chain complexes, and α, β, γ are chain maps. Then the following diagram commutes as well:

$$\begin{tikzcd}
 & c_{p-1} & d_p & e_p & & \\
\cdots & H_p(\mathcal{C}) \arrow[r, "\phi_*"] \arrow[d, "\alpha_*"'] & H_p(\mathcal{D}) \arrow[r, "\psi_*"] \arrow[d, "\beta_*"'] & H_p(\mathcal{E}) \arrow[r, "\partial_*"] \arrow[d, "\gamma_*"', "e'_p"] & H_{p-1}(\mathcal{C}) \arrow[r] \arrow[d, "\alpha_*"'] & \cdots \\
\cdots & H_p(\mathcal{C}') \arrow[r, "\phi'_*"] & H_p(\mathcal{D}') \arrow[r, "\psi'_*"] & H_p(\mathcal{E}') \arrow[r, "\partial'_*"] & H_{p-1}(\mathcal{C}') \arrow[r] & \cdots
\end{tikzcd}$$

Notice that each "level" here, e.g., $0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$, represents a collection of groups and homomorphisms as we have seen previously. We have exactness within this substructure, and similarly within the $\mathcal{C}', \mathcal{D}', \mathcal{E}'$ substructure. α, β, γ are chain maps connecting corresponding parts of the two substructures.

Proof Commutativity of first and second squares is immediate, as it holds at the chain level. Commutativity of the last (3rd) square involves the definition of ∂_* and ∂'_*. Given $\{e_p\} \in H_p(\mathscr{E})$, choose d_p such that $\psi(d_p) = e_p$, and choose c_{p-1} such that $\phi(c_{p-1}) = \partial_D d_p$. Then $\partial_* \{e_p\} = \{c_{p-1}\}$ by definition.

Notice that we are not explicitly displaying this "internal" zig-zag in the picture above. Now we want to consider the corresponding images under γ, β, and α, and show that the structure is "preserved".

Let $e_p' = \gamma(e_p)$; we want to show $\partial'_* \{e'_p\} = \alpha_* \{c_{p-1}\}$.

Intuitively, this result follows because each step in the definition of ∂_* commutes.

$$\begin{tikzcd}
0 \arrow[r] & \overset{c_{p-1}}{\mathscr{C}} \arrow[r, "\phi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\square_2"] & \overset{d_p}{\mathscr{D}} \arrow[r, "\psi"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\square_1"] & \overset{e_p}{\mathscr{E}} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \underset{c_{p-1}}{\mathscr{C}'} \arrow[r, "\phi'"] & \underset{d_p'}{\mathscr{D}'} \arrow[r, "\psi'"] & \underset{e_p'}{\mathscr{E}'} \arrow[r] & 0
\end{tikzcd}$$

$\beta(d_p)$ is a suitable pullback for e_p', as $\square_1$ commutes:

$$\psi' \beta(d_p) = \gamma \psi(d_p) = \gamma(e_p) = e_p'.$$

Similarly, $\alpha(c_{p-1})$ is a suitable pullback for $\partial_D' \beta(d_p)$, since $\square_2$ commutes:

$$\phi' \alpha(c_{p-1}) = \beta \phi(c_{p-1}) = \beta(\partial_D d_p) = \partial_D'(\beta(d_p)).$$

$\Longrightarrow$ $\partial'_* \{e_p'\} = \{\alpha(c_{p-1})\}$ by definition. $\square$

Here is another result in the same flavor.

Lemma 24.3 [M] (**The Steenrod five lemma**) Suppose we are given the commutative diagram of abelian groups and homomorphisms

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

where the horizontal sequences are exact. If f_1, f_2, f_4, f_5 are all isomorphisms, then so is f_3.

You'll get a chance to prove this lemma in homework ☺!

Application to relative homology: see Lemma 24.4 and Theorem 24.5 in [M].

Meyer-Vietoris Sequences

We use the zig-zag lemma to derive another long exact sequence to compute homology groups. It relates the homology of two given spaces to that of their union and their intersection. The overarching theme is once again the "easy" or "efficient" identification or computation of homology groups.

Theorem 25.1 [M] Let K be a complex, and $K', K'' \subseteq K$ be subcomplexes such that $K = K' \cup K''$. Let $A = K' \cap K''$. Then there is a long exact sequence

$$\cdots H_p(A) \longrightarrow H_p(K') \oplus H_p(K'') \longrightarrow H_p(K) \xrightarrow{\partial} H_{p-1}(A) \rightarrow \cdots$$

called the Meyer-Vietoris sequence of (K', K''). There exists a similar exact sequence in reduced homology if A is nonempty.

∂ is the connecting homomorphism — notice that ∂ takes us from dimension p to $p-1$.

Notation: The book uses different notation. The one used here is probably more intuitive. We will use $'$ and $''$ as superscripts for all objects related to K' and K'', respectively.

Proof idea: We construct short exact sequences of chain complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K') \oplus \mathcal{C}(K'') \xrightarrow{\psi} \mathcal{C}(K) \longrightarrow 0$$

and apply the zig-zag lemma.