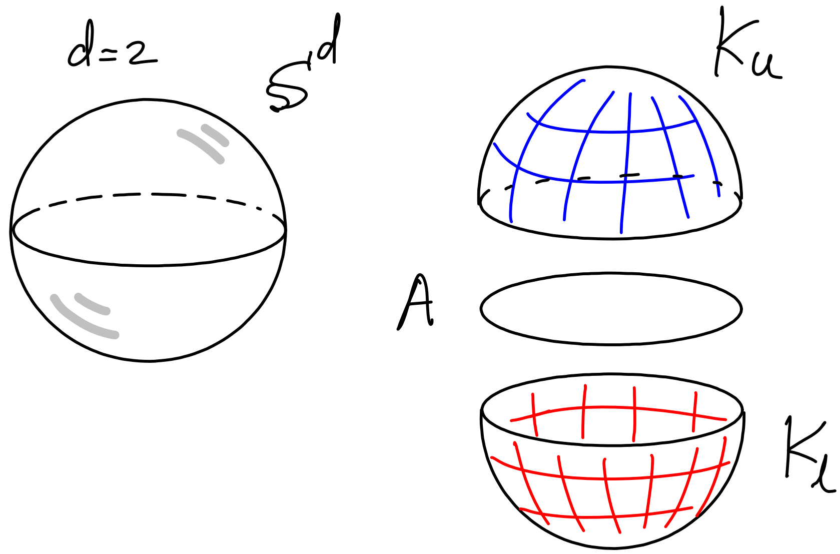
MATH 524: Lecture 22 (10/30/2025)

Today: * Applications of MVS

Recall: Homology of $\mathbb{S}^d$: $\tilde{H}_p(\mathbb{S}^d) \simeq \mathbb{Z}$ if $p=d$, $=0$ o.w.



Let's consider absolute homology now. The MVS is

$$\cdots \to H_p(\mathbb{S}^{d-1}) \xrightarrow{i_*} H_p(K_u) \oplus H_p(K_\ell) \xrightarrow{j_*} H_p(\mathbb{S}^d) \xrightarrow{\partial_*}$$

$$H_{p-1}(\mathbb{S}^{d-1}) \xrightarrow{i_*} H_{p-1}(K_u) \oplus H_{p-1}(K_\ell) \xrightarrow{j_*} H_{p-1}(\mathbb{S}^d) \to$$

$$\to \cdots$$

In the middle, again, we get for $p>0$,

$$0 \oplus 0 \longrightarrow H_p(\mathbb{S}^d) \longrightarrow H_{p-1}(\mathbb{S}^{d-1}) \longrightarrow 0 \oplus 0 .$$

Notice that $H_p(K_u)$ and $H_p(K_\ell)$ are both trivial for $p>0$, as we had with reduced homology groups (as they are both balls). Thus the middle map is an isomorphism. We will use this general result in the induction!

Here are the details for $d=2$ about how we finish. Arguments are similar for more general d.

$$0 \oplus 0 \longrightarrow H_2(\mathbb{S}^2) \longrightarrow$$

$$\longrightarrow \underset{\mathbb{Z}}{H_1(\mathbb{S}^1)} \xrightarrow{i_*} \overset{0}{H_1(K_u)} \oplus \overset{0}{H_1(K_\ell)} \xrightarrow{j_*} H_1(\mathbb{S}^2) \longrightarrow$$

$$\longrightarrow \underset{\mathbb{Z}}{H_0(\mathbb{S}^1)} \xrightarrow{i_*} \underset{\mathbb{Z}}{H_0(K_u)} \oplus \underset{\mathbb{Z}}{H_0(K_\ell)} \xrightarrow{j_*} H_0(\mathbb{S}^2) \longrightarrow 0.$$

single component each

We can look at smaller portions of the sequence to figure out the structure of the homology groups we seek.

First part: $\quad 0 \longrightarrow H_2(\mathbb{S}^2) \xrightarrow{\partial_*} \mathbb{Z} \longrightarrow 0$ $\;$ $0 \oplus 0$, to be precise

$\Rightarrow \partial_*$ is an isomorphism $\Rightarrow H_2(\mathbb{S}^2) \simeq \mathbb{Z}$.

Second part:

$$0 \oplus 0 \longrightarrow H_1(\mathbb{S}^{1^2}) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_*} H_0(\mathbb{S}^{1^2}) \longrightarrow 0$$

$\uparrow \mathbb{Z}$, by exactness

Let's look at the structure of $i_\#$. Notice that $i_\#$ is injective, and $\ker i_\# = 0$. By exactness, we get $\operatorname{im} \partial_* = \ker i_\# = 0$, which gives that $H_1(\mathbb{S}^{1^2}) = 0$.

a 0-chain in K_u or K_ℓ corresponds injectively to the 0-chain in $A = \mathbb{S}^1$.

Then we could apply induction to get the result:

$H_p(\mathbb{S}^{1^d}) \simeq \mathbb{Z}$ when $p = d$ or $p = 0$, and

$H_p(\mathbb{S}^{1^d}) = 0$ otherwise. $\square$

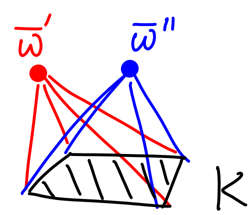
Example 2 Homology of the suspension of a simplicial complex.

Def Given a simplicial complex K, let $\bar{w}' * K$ and $\bar{w}'' * K$ be two cones whose polytopes intersect in $|K|$ alone. Then $S(K) = (\bar{w}' * K) \cup (\bar{w}'' * K)$ is a complex called the **suspension** of K. $S(K)$ is uniquely defined up to simplicial isomorphism.

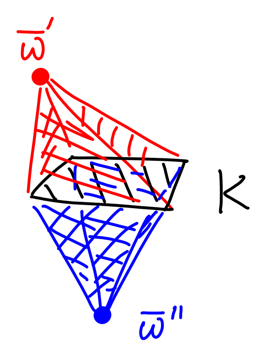
Recall the definition of cone from Lecture 16:

Def Let K be a simplicial complex in $\mathbb{R}^d$, and $\bar{w} \in \mathbb{R}^d$ is a point such that each ray emanating from $\bar{w}$ intersects $|K|$ in at most one point. Then the **cone of K with vertex $\bar{w}$** is the collection of all simplices of the form $\bar{w}a_0 \dots a_p$ where $\bar{a}_0 \dots \bar{a}_p$ is a simplex of K, along with all faces of such simplices. We denote this collection as $\bar{w} * K$.

Indeed, the specific choices of $\bar{w}'$ and $\bar{w}''$ are not important, due to the restriction that the two cones intersect only in $|K|$. Thus we do not get the situation shown here, where the two cones intersect outside of $|K|$.



Due to the same intersection condition, it would also follow that $\bar{w}'$ and $\bar{w}''$ are on the "opposite sides" of K. Hence the name suspension is quite appropriate — K is "suspended" in the middle by connections from $\bar{w}'$ and $\bar{w}''$.



We want to study how $H_*(S(K))$ and $H_*(K)$ are related. And we will use the Mayer-Vietoris sequence in a natural way.

Theorem 25.4 [M] For a simplicial complex K, there is an isomorphism $\tilde{H}_p(S(K)) \longrightarrow \tilde{H}_{p-1}(K)$ $\forall p$.

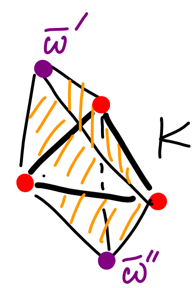
<u>Proof</u> Let $K' = \bar{\omega}' * K$, $K'' = \bar{\omega}'' * K$. Then $K' \cup K'' = S(K)$, and $A = K' \cap K'' = K$. In the reduced homology Meyer-Vietoris sequence, we have

$$\underbrace{\tilde{H}_p(K')}_{0} \oplus \underbrace{\tilde{H}_p(K'')}_{0} \xrightarrow{j_\#} \tilde{H}_p(S(K)) \xrightarrow{\partial_*} \tilde{H}_{p-1}(\overset{A}{K}) \longrightarrow \underbrace{\tilde{H}_{p-1}(K')}_{0} \oplus \underbrace{\tilde{H}_{p-1}(K'')}_{0}$$

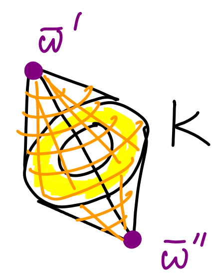
Both end terms vanish $(0 \oplus 0)$ as K', K'' are both cones.

Hence the middle map is an isomorphism. $\square$

Here is an example. Let K consist of 3 edges and 3 vertices forming a circle ($\approx S^1$). Then $S(K)$ consists of 6 triangles forming the surface of a sphere. Indeed, $S(K) \approx S^2$ and we do have $H_2(S(K)) \simeq H_1(K) \simeq \mathbb{Z}$.



A bit more interesting version of this example has K an annulus. Then both K' and K'' are solid 3D "half cones", with $S(K)$ enclosing a single void in between.

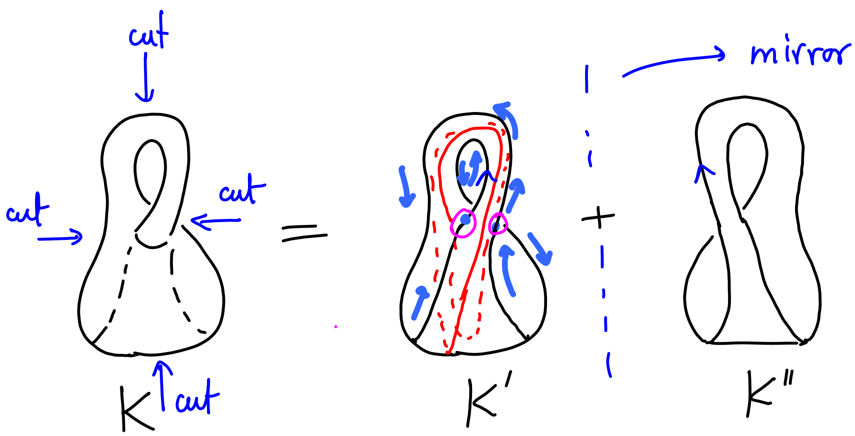


We can naturally talk about $S(S(K))$, which is the suspension of a suspension of K which is also denoted $S^2(K)$.

We could consider $S(K)$ also in the abstract setting.

Example 3 Klein bottle

We now consider the homology of $\mathbb{K}^2$ using its Mayer-Vietoris sequence. Imagine cutting the Klein bottle down the middle into two pieces, both of which are Möbius strips. We denote the original object/space by K, and the two pieces by K' and K''. We get K by gluing K' and K'' along the "cut", i.e., along the edges of the two Möbius strips.

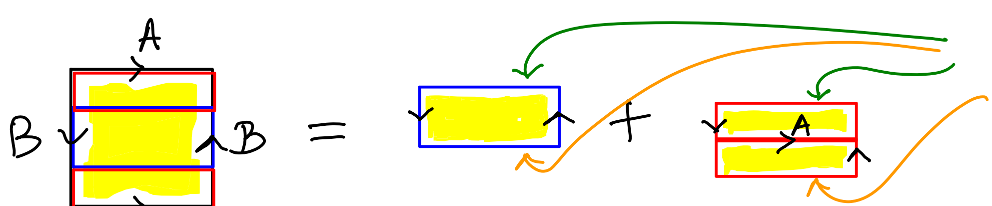


cut K down the middle

Here's a more illustrative picture:

(image: www)

We could also represent the splitting on the square diagram (with pairs of opposite edges identified appropriately).



glue along the "edge" of each Möbius strip

two Möbius strips

Another way to consider the Klein bottle is to imagine cutting out 2 disks from a 2-sphere, and gluing 2 Möbius strips along the boundaries created by the cuts, which are circles.

Thus we have $\mathbb{K}^2 \approx K = K' \cup K''$; $A = K' \cap K'' \approx S^1$; K', K'' are both Möbius strips.

Let's consider the reduced homology Meyer-Vietoris sequence:

$$0 \longrightarrow \overset{0}{\tilde{H}_2(A)} \xrightarrow{i_\#} \overset{0}{\tilde{H}_2(K')} \oplus \overset{0}{\tilde{H}_2(K'')} \xrightarrow{\partial_\#} \tilde{H}_2(K)$$

$$\longrightarrow \underset{\mathbb{Z}}{\tilde{H}_1(A)} \xrightarrow{i_\#} \underset{\mathbb{Z}}{\tilde{H}_1(K')} \oplus \underset{\mathbb{Z}}{\tilde{H}_1(K'')} \xrightarrow{\partial_\#} \tilde{H}_1(K)$$

$$\longrightarrow \underset{0}{\tilde{H}_0(A)}$$

Notice $A \approx S^1$, hence $\tilde{H}_1(A) \simeq \mathbb{Z}$. Similarly, since K' and K'' are both Möbius strips, $\tilde{H}_1(K') \simeq \mathbb{Z}$ and $\tilde{H}_1(K'') \simeq \mathbb{Z}$.

We will finish the argument in the next lecture...