MATH 524: Lecture 23 (11/04/2025)

Today: * MVS of $\mathbb{K}^2$
* MVS of $\mathbb{T}^2$
* categories

Recall : $\mathbb{K}^2 \simeq K = K' \cup K''$, $A = K' \cap K''$

Möbius strips circle

$$\begin{tikzcd}
0 \arrow[r] & \underset{}{\tilde{H}_2(A)} \arrow[r, "i_*"] & \overset{0}{\tilde{H}_2(K')} \oplus \overset{0}{\tilde{H}_2(K'')} \arrow[r, "j_*"] & \tilde{H}_2(K) ? \\
 & \underset{\mathbb{Z}}{\tilde{H}_1(A)} \arrow[r, "i_*"] & \underset{\mathbb{Z}}{\tilde{H}_1(K')} \oplus \underset{\mathbb{Z}}{\tilde{H}_1(K'')} \arrow[r, "\partial_*"] & \tilde{H}_1(K) ? \\
 & \underset{0}{\tilde{H}_0(A)} & &
\end{tikzcd}$$

Recall that $H_1(A) = H_1(S^1) \simeq \mathbb{Z}$; similarly, $H_1(K') \simeq H_1(K'') \simeq \mathbb{Z}$, as they are both Möbius strips. H_2 is trivial in all cases. Also, $\tilde{H}_0(A) = 0$, as it has one component.

First piece: $0 \longrightarrow \tilde{H}_2(K) \xrightarrow{\partial_*} \underset{A}{\mathbb{Z}} \xrightarrow{i_*} \underset{K'}{\mathbb{Z}} \oplus \underset{K''}{\mathbb{Z}} \longrightarrow \cdots$

which induces i_*

Let's consider $i_\#$. It's given by $i_\# : 1 \longrightarrow (2,-2)$. Notice that the "edge" of a Möbius strip wraps twice around its "middle" circle. Also, the two Möbius strips K' and K'' are mirror images, so to speak. In particular, the orientations of their "edges" are opposite. Hence the map is given as $(2,-2)$. We note that $i_\#$ is injective (every cycle in $\tilde{H}_1(K')$ and $\tilde{H}_1(K'')$ correspond uniquely to a cycle in $\tilde{H}_1(A)$, up to homology). Also, we note that $\ker i_* = 0$. So $\operatorname{im} \partial_* = 0$, due to exactness at $\tilde{H}_1(A)$.

Hence $\tilde{H}_2(K) = 0$.

To identify $\tilde{H}_1(K)$, we look at the second piece:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_*} \tilde{H}_1(K) \xrightarrow{\partial_*} 0$$

We can apply Result 3 on exact sequences (Lecture 17) to get that $\dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\operatorname{im} i_*} \simeq \tilde{H}_1(K)$.

3. Suppose the sequence $0 \to A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \to 0$ is exact; then $A_2/\phi(A_1) = \operatorname{cok} \psi$ is isomorphic to A_3; this isomorphism is induced by ψ.

First, note that $i_* : 1 \longrightarrow (2,-2)$, i.e., $\operatorname{im} i_* = 2\mathbb{Z}\begin{pmatrix}1\\-1\end{pmatrix}$ or $\mathbb{Z}\begin{pmatrix}2\\-2\end{pmatrix}$.

One basis for $\mathbb{Z} \oplus \mathbb{Z}$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ $\left(\text{as } \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\1\end{pmatrix}\text{, where } \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \text{ is a natural basis for } \mathbb{Z} \oplus \mathbb{Z}\right)$.

So $\dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\operatorname{im} i_*} \simeq \mathbb{Z} \oplus \mathbb{Z}/2$.

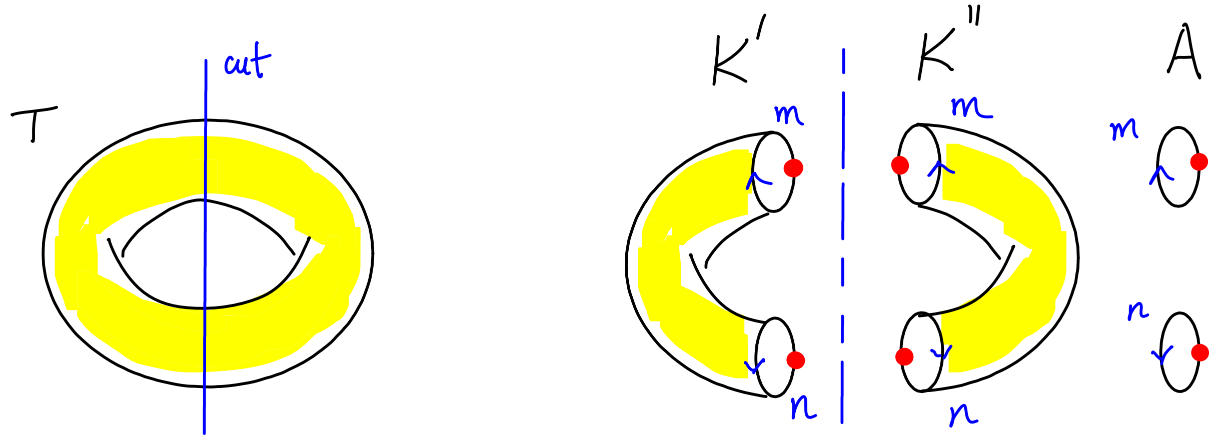
Hence $\tilde{H}_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$.

Using $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ as the basis is motivated by $\operatorname{im} i_*$ being $(\simeq)$ $2\mathbb{Z}\begin{pmatrix}1\\-1\end{pmatrix}$. With this basis, we can perform the quotienting directly.

Example 4: Torus

We split the torus down the middle into two cylinders whose intersection is the union of two disjoint circles.

We consider the Meyer-Vietoris sequence in absolute homology.



K', K'': cylinders : $H_2(K'') = 0$, $H_1(K'') \simeq \mathbb{Z}$, $H_0(K'') \simeq \mathbb{Z}$.

A : two disjoint circles: $H_1(A) \simeq \mathbb{Z}\oplus\mathbb{Z}$, $H_0(A) \simeq \mathbb{Z}\oplus\mathbb{Z}$.

$$0 \longrightarrow H_2(T) \xrightarrow{\partial_*}$$

$$\overset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A)} \xrightarrow{i_*} \overset{\mathbb{Z}}{H_1(K')} \oplus \overset{\mathbb{Z}}{H_1(K'')} \xrightarrow{j_*} H_1(T) \xrightarrow{\partial_*}$$

$$\underset{\mathbb{Z}\oplus\mathbb{Z}}{H_0(A)} \xrightarrow{i_*} \underset{\mathbb{Z}}{H_0(K')} \oplus \underset{\mathbb{Z}}{H_0(K'')} \xrightarrow{j_*} \underset{\mathbb{Z}}{H_0(T)}$$

we assume $H_0(T) \simeq \mathbb{Z}$, as it has one component.

First piece:

$$0 \longrightarrow H_2(T) \xrightarrow{\partial_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A)} \xrightarrow{i_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(K') \oplus H_1(K'')}$$

i_* maps (m,n) to $(m-n, -m+n)$. Notice that $\ker i_* \simeq \mathbb{Z}$ (we get $(0,0)$ when $m=n$). By exactness at $H_1(A)$, we get $\operatorname{im} \partial_* = \ker i_* \simeq \mathbb{Z}$. Also, ∂_* is injective (see Rule 2 from Lecture 17). Hence $H_2(T) \simeq \mathbb{Z}$.

a set of generators, not a basis

Notice that $\operatorname{im} i_* \simeq \mathbb{Z}$ $\left(\mathbb{Z}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}\right\},\ \text{but } \begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix}.\right)$

More directly, $m-n$ and $-m+n$ are not independent of each other.

The inclusion homomorphism i_* at level 0 has identical structure to the i_* at level 1. i_* again maps (m,n) to $(m-n, -m+n)$. Consider two points, one each in the 2 circles in A, with multipliers m, n, respectively, and how i_* maps them to K' and K''.

two points, one on either circular boundary of the cylinder, are homologous due to a 1-chain connecting them (on the wall of the cylinder).

Second piece: To identify $H_1(T)$, we consider five groups in the sequence with $H_1(T)$ in the middle.

$$H_1(A) \xrightarrow{i_*} H_1(K') \oplus H_1(K'') \xrightarrow{j_*} H_1(T) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(K') \oplus H_0(K'')$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_*} H_1(T) \xrightarrow{\partial_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z}$$

Use Result 5 on exact sequences (Lee 17):

5. Suppose the sequence $A_1 \xrightarrow{\alpha} A_2 \to A_3 \to A_4 \xrightarrow{\beta} A_5$ is exact. Then so is the induced sequence $0 \to \operatorname{cok} \alpha \to A_3 \to \ker \beta \to 0$.

So,

$0 \longrightarrow \operatorname{cok} i_* \longrightarrow H_1(T) \longrightarrow \ker i_* \longrightarrow 0$ is exact.

$\operatorname{im} i_* \simeq \mathbb{Z}$, so $\operatorname{cok} i_* \simeq \dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} = \mathbb{Z}$.

$\Rightarrow \quad 0 \to \mathbb{Z} \to H_1(T) \to \mathbb{Z} \to 0$ is exact.

$\Rightarrow \quad H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

In the last few lectures, we will give a brief overview of cohomology, which is "dual" to homology. The concepts used to define cohomology are lot more algebraic in nature. We start by introducing the machinery of categories and functors.

Categories and Functors

§28 in [M]

[M] defines three things; which we list here as 1, 2, 3 as well

Def A **category** $\mathcal{C}$ consists of two things:

1. A class (or a collection) of **objects** $(\mathcal{C}_o)$; "oh"
2. for every ordered pair (X, Y) with $X, Y \in \mathcal{C}_o$, a set **hom(X,Y)** of **morphisms** f (or arrows).
   One writes $f: X \to Y$ or $X \xrightarrow{f} Y$ for the morphism $f \in \hom(X,Y)$. Here, $X = \mathrm{dom}(f)$, i.e., its domain, and $Y = \mathrm{cod}(f)$, i.e., its codomain.
   The collection of all morphisms is denoted $\mathcal{C}_m$. this is the second "thing".
3. A function, called the **composition of morphisms** is defined for every triple (X, Y, Z) of objects:
   $$\hom(X,Y) \times \hom(Y,Z) \longrightarrow \hom(X,Z).$$
   The image of the pair (f, g) under composition is defined as $g \circ f$ (or gf).

The second "thing" $\mathcal{C}_m$ must have the compositions defined — the book calls this the third "thing".

In other words, when we have morphisms f and g with $\mathrm{dom}(f) = \mathrm{cod}(g)$, the composition of f and g is gf with its domain as $\mathrm{dom}(f)$ and codomain as $\mathrm{cod}(g)$.

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (X \xrightarrow{gf} Z)$$

The following two properties must be satisfied by the objects.

4. **Axiom 1** (Associativity) The composition of morphisms is associative: If $f \in \hom(W,X)$, $g \in \hom(X,Y)$, $h \in \hom(Y,Z)$, then
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
Another notation: If $f: W \to X$, $g: X \to Y$, $h: Y \to Z$, then $h(gf) = (hg)f$.

5. **Axiom 2** (Existence of identity) some other books use id_X
For every $X \in \mathcal{C}_0$, there exists a morphism $1_X \in \hom(X,X)$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ $\forall f \in \hom(W,X)$ and $g \in \hom(X,Y)$, where W and Y are arbitrary objects.

Notice that 1_X (identity morphism) is unique. Suppose $1_X \circ f = f$ and $g = g \circ 1'_X$ $\forall f \in \hom(W,X)$ and $\forall g \in \hom(X,Y)$.
(we are assuming there exist two identity morphisms $1_X, 1'_X$).
Then, setting $f = 1'_X$ and $g = 1_X$, we get
$$1_X \circ 1'_X = 1'_X \quad \text{and} \quad 1_X = 1_X \circ 1'_X, \quad \text{i.e.,} \quad 1_X = 1'_X.$$

Examples of categories

1. $\bar{1}$: a category with one object $*$ and one morphism 1_*.
2. Top : The category of topological spaces and continuous maps.
3. Grp : The category of groups and group homomorphisms.