MATH 524: Lecture 24 (11/06/2025)

Today: more on categories and functors

Recall: Category: $\mathcal{C}_o$, $\mathcal{C}_m$, Axiom 1: $h(gf) = (hg)f$; Axiom 2: $1_X \circ f = f$, $f \circ 1_X = f$.

Examples of categories

1. $\mathcal{I}$: a category with one object $*$ and one morphism 1_*.
2. Top: The category of topological spaces and continuous maps.
3. Grp: The category of groups and group homomorphisms.
4. Set: The category of sets and functions between sets.
5. Simplicial complexes and simplicial maps
6. Chain complexes and chain maps.
7. Short exact sequences and homomorphisms between them.

We introduce one more concept on categories, and then introduce functors as maps between categories.

Def (Inverse) Let $f \in \hom(X,Y)$, and $g, g' \in \hom(Y,X)$.

If $g \circ f = 1_X$, we call g a **left inverse** of f. If $f \circ g' = 1_Y$, we call g' a **right inverse** of f

$$X \underset{g,g'}{\overset{f}{\rightleftarrows}} Y$$

If f has a left inverse g and a right inverse g', then they are equal. Since $(g \circ f) \circ g' = 1_X \circ g' = g'$ and $g \circ (f \circ g') = g \circ 1_Y = g$, and hence by Axiom 1, $g = g'$. This map $g = g'$ is called an **inverse** to f, and it is unique.

If f has an inverse, it is called an **equivalence**.

In example 5. Simplicial complexes and simplicial maps, the equivalences are simplicial homeomorphisms.

Def A (covariant) **functor** F from category $\mathbb{C}$ to category $\mathbb{D}$ is a function assigning to each object X of $\mathbb{C}$ an object $F(X)$ of $\mathbb{D}$, and to each morphism $f: X \to Y$ of $\mathbb{C}$ the morphism $F(f): F(X) \longrightarrow F(Y)$ of $\mathbb{D}$ such that

$$F(1_X) = 1_{F(X)} \quad \forall\, X \in \mathbb{C}_0 \quad \text{and}$$

$$F(g \circ f) = F(g) \circ F(f) \quad \forall\, f, g \in \mathbb{C}_m.$$

Here is a pictorial representation:

$$\left\{ \begin{array}{ccc} \mathbb{C} & & X \xrightarrow{f} Y \xrightarrow{g} Z \\ F \downarrow & & \\ \mathbb{D} & & F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \end{array} \right.$$

Informally, a functor preserves composition and identities. Also, if f is an equivalence in $\mathbb{C}$, then $F(f)$ is an equivalence in $\mathbb{D}$.

Examples

1. There is a functor $\mathcal{U}: \mathrm{Top} \to \mathrm{Set}$ which assigns to any topological space X its underlying set (i.e, $|X|$ as a set), and to continuous maps their underlying set maps. This functor is called the **forgetful functor**, as it "forgets" the topological structure involved.

Indeed, the functor $\mathcal{U}$ here does not preserve all the "structure"!

2. The correspondence $K \to \mathcal{C}(K)$ and $f \to f_\#$ is a functor from the category of simplicial complexes and simplicial maps to the category of chain complexes and chain maps.

3. The zig-zag lemma assigns to each short exact sequence of chain complexes a long exact sequence of homology groups.

The functors we have seen so far "preserve" the arrows. We can also define functors that "reverse" each arrow.

Def A **contravariant functor** G from category $\mathbb{C}$ to category $\mathbb{D}$ is a rule that assigns to each object X of $\mathbb{C}$ an object $G(X)$ of $\mathbb{D}$, and to each morphism $f: X \to Y$ of $\mathbb{C}$ a morphism $G(f): G(Y) \longrightarrow G(X)$ of $\mathbb{D}$, such that

$$G(1_X) = 1_{G(X)} \quad \text{and}$$

$$G(g \circ f) = G(f) \circ G(g).$$

notice that the arrow is reversed here!

We can define the "opposite" category $\mathbb{C}^{op}$ for a given category $\mathbb{C}$ by setting $\mathbb{C}_0^{op} = \mathbb{C}_0$ but the morphisms are reversed. If $f: X \to Y \in \mathbb{C}_m$, then $f^{op}: Y \to X \in \mathbb{C}_m^{op}$.

Composition is defined as: $f^{op} \circ g^{op} = (g \circ f)^{op}$.

Then, a contravariant functor G from $\mathbb{C}$ to $\mathbb{D}$ is a (covariant) functor from $\mathbb{C}^{op}$ to $\mathbb{D}$, or equivalently from $\mathbb{C}$ to $\mathbb{D}^{op}$!

Notice that f^{op} need not be the "inverse" of f, at least not in general. All we require is that the direction of the morphism is reversed. In specific examples, f^{op} could be equal to f^{-1}, though.

More examples of functors

4. Given two categories $\mathbb{C}, \mathbb{D}$, we can define a product category $\mathbb{C} \times \mathbb{D}$, which has as objects the pairs $(C, D) \in \mathbb{C}_0 \times \mathbb{D}_0$, and as morphisms from (C, D) to (C', D') the pairs (f, g) with $f: C \to C' \in \mathbb{C}_m$, and $g: D \to D' \in \mathbb{D}_m$, denoted $f \times g$. Then there are the **projection functors** $\pi_{\mathbb{C}}: \mathbb{C} \times \mathbb{D} \to \mathbb{C}$ and $\pi_{\mathbb{D}}: \mathbb{C} \times \mathbb{D} \to \mathbb{D}$.

5. Given functors $F: \mathbb{C} \to \mathbb{D}$ and $G: \mathbb{D} \to \mathbb{E}$, we can define the composition functor $GF: \mathbb{C} \to \mathbb{E}$.
This composition operation is associative (use F, G, H).
There is also the identity functor of every category:
$1: \mathbb{C} \to \mathbb{C}$. So we get Cat, the category which has a objects categories, and as morphisms the functors.

We introduce one more concept on how to combine multiple functors to define more general objects.

Def Let G, H be two functors from $\mathbb{C}$ to $\mathbb{D}$. A **natural transformation** T from G to H is a rule assigning to each object X of $\mathbb{C}$ a morphism

$$T_X : G(X) \longrightarrow H(X) \text{ of } \mathbb{D}$$

such that the following diagram commutes for every morphism $f: X \to Y$ of $\mathbb{C}$.

$$\begin{tikzcd} G(X) \arrow[r, "T_X"] \arrow[d, "G(f)"] & H(X) \arrow[d, "H(f)"] \\ G(Y) \arrow[r, "T_Y"] & H(Y) \end{tikzcd}$$

If for each X, the morphism T_X is an equivalence in $\mathbb{D}$, then T is called a **natural equivalence** of the functors.