MATH 524: Lecture 25 (11/13/2025)

Today: * Hom functor
* cohomology groups of simplicial complexes

Cohomology

The Hom functor

§41 in [M]

Def Let A, G be abelian groups. Then the set $\text{Hom}(A,G)$ of all homomorphisms from A to G becomes an abelian group if we add two homomorphisms by adding their values in G.

$\phi, \psi: A \to G$ are homomorphisms

For $a \in A$, we define $(\phi+\psi)(a) = \phi(a)+\psi(a)$.

The map $\phi+\psi$ is a homomorphism, as

$(\phi+\psi)(0)=0$ and

$$
\begin{aligned}
(\phi+\psi)(a+b) &= \phi(a+b) + \psi(a+b) \\
&= \phi(a)+\psi(a) + \phi(b)+\psi(b) \\
&= (\phi+\psi)(a) + (\phi+\psi)(b).
\end{aligned}
$$

The identity element of $\text{Hom}(A,G)$ is the homomorphism mapping A to id_G (or 1_G), the identity element of G.

The inverse of homomorphism $\phi: A \to G$ is the homomorphism that maps a to $-\phi(a)$ $\forall a \in A$.

Example $\text{Hom}(\mathbb{Z},G)$ is isomorphic to G itself. The isomorphism assigns to the homomorphism $\phi: \mathbb{Z} \to G$ the element $\phi(1)$.

Notice that any homomorphism $\phi: \mathbb{Z} \to G$ is completely determined by $\phi(1)$.

More generally, if A is free-abelian with finite rank and basis $e_1, \dots, e_n$, then $\text{Hom}(A,G)$ is isomorphic to $\underbrace{G \oplus \cdots \oplus G}_{n \text{ copies}}$.

This isomorphism assigns to any homomorphism $\phi: A \to G$ the n-tuple $(\phi(e_1), \dots, \phi(e_n))$.

As the name cohomology suggests, we want define objects that are dual to homology. Indeed, we define homomorphisms from $\mathrm{Hom}(B,G)$ to $\mathrm{Hom}(A,G)$ for given homomorphisms from $A \longrightarrow B$.

Def A homomorphism $f: A \to B$ gives rise to a **dual homomorphism**

$$\mathrm{Hom}(A,G) \xleftarrow{\tilde{f}} \mathrm{Hom}(B,G)$$

going in the reverse direction.

The map $\tilde{f}$ assigns to the homomorphism $\phi: B \to G$, the composite $A \xrightarrow{f} B \xrightarrow{\phi} G$. That is, $\boldsymbol{\tilde{f}(\phi) = \phi \circ f}$.

$\tilde{f}$ is indeed a homomorphism, as $\tilde{f}(0) = 0$, and

$$\begin{aligned} [\tilde{f}(\phi+\psi)](a) &= (\phi+\psi)(f(a)) = \phi(f(a)) + \psi(f(a)) \\ &= [\tilde{f}(\phi)](a) + [\tilde{f}(\psi)](a). \end{aligned}$$

For a fixed G, the assignment $A \to \mathrm{Hom}(A,G)$ and $f \to \tilde{f}$ defines a contravariant functor from the category of abelian groups and homomorphisms to itself.

Recall: The opposite category: $\mathcal{C}^{op}$.

Given category $\mathcal{C}$, we consider another category $\mathcal{C}^{op}$ with $\mathcal{C}^{op}_0 = \mathcal{C}_0$ (same objects), but with morphisms reversed: so, if $f : X \to Y \in \mathcal{C}_m$, then $f^{op} : Y \to X \in \mathcal{C}^{op}_m$.

Composition: $f^{op} g^{op} = (gf)^{op}$.

Then, a contravariant functor G from $\mathbb{C}$ to $\mathbb{D}$ is a (covariant) functor from $\mathbb{C}^{op}$ to $\mathbb{D}$, or equivalently from $\mathbb{C}$ to $\mathbb{D}^{op}$.

For, if $i_A : A \to A$ is the identity homomorphism, then $\tilde{i}_A$ is the identity map of $Hom(A, G)$. $\tilde{i}_A(\phi) = \phi \circ i_A = \phi$. Hence $\tilde{i}_A$ is the identity map of $Hom(A,G)$.

Also, if the left diagram commutes, so does the right one.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} Hom(A,G) & & Hom(C,G) \arrow[ll, "\tilde{h}"'] \arrow[dl, "\tilde{g}"] \\ & Hom(B,G) \arrow[ul, "\tilde{f}"] & \end{tikzcd}$$

For, $\tilde{h}(\phi) = \phi \circ h = \phi \circ (g \circ f)$, as left diagram commutes.

and $\tilde{f}(\tilde{g}(\phi)) = \tilde{f}(\phi \circ g) = (\phi \circ g) \circ f$, which are equal.

We state a few implications of this correspondence. There are many more results listed in the book. We will then use $\mathrm{Hom}(C_p(K), G)$ to define cohomology groups.

Theorem 41.1 [M] Let f be a homomorphism, and $\tilde{f}$ its dual homomorphism.

(a) If f is an isomorphism, so is $\tilde{f}$.

(b) If f is the zero homomorphism, so is $\tilde{f}$.

(c) If f is surjective, then $\tilde{f}$ is injective. So the exactness of $B \xrightarrow{f} C \longrightarrow 0$ implies the exactness of

$$\mathrm{Hom}(B,G) \xleftarrow{\tilde{f}} \mathrm{Hom}(C,G) \longleftarrow 0.$$

Proof (c) f is surjective. Let $\psi \in \mathrm{Hom}(C,G)$ and suppose $\tilde{f}(\psi) = 0 = \psi \circ f$. So $\psi(f(b)) = 0 \;\forall\; b \in B$. Since f is surjective, we get that $\psi(c) = 0 \;\forall\; c \in C$.

Simplicial Cohomology Groups

Def Let K be a simplicial complex, G be an abelian group. The **group of p-dimensional cochains** of K with coefficients in G is the group $C^p(K;G) = \mathrm{Hom}(C_p(K), G)$. The **coboundary operator** δ^p is defined as the dual of the boundary operator $\partial_{p+1} : C_{p+1}(K) \longrightarrow C_p(K)$. Thus

$$C^{p+1}(K;G) \xleftarrow{\ \delta^p\ } C^p(K;G).$$

So δ raises dimension by 1. We define $Z^p(K;G) = \ker \delta^p$ and $B^{p+1}(K;G) = \mathrm{im}\, \delta^p$, the groups of p-cocycles and (p+1)-coboundaries with coefficients in G. We take $G = \mathbb{Z}$ as the default choices.

If $\bar{c}_p$ is a p-chain, $\bar{c}_p \in C_p$, and ϕ^p is a p-cochain, $\phi^p \in C^p$, then the cochain ϕ^p evaluates $\bar{c}_p$ by mapping it to $\mathbb{Z}$. We denote this evaluation by $\phi^p(\bar{c}_p) = \langle \phi^p, \bar{c}_p \rangle$. ← this notation is preferred

We get $\langle \delta\phi^p, \bar{d}_{p+1} \rangle = \langle \phi^p, \partial \bar{d}_{p+1} \rangle$, or more generally,

$$\langle \delta\phi, \bar{c} \rangle = \langle \phi, \partial \bar{c} \rangle.$$

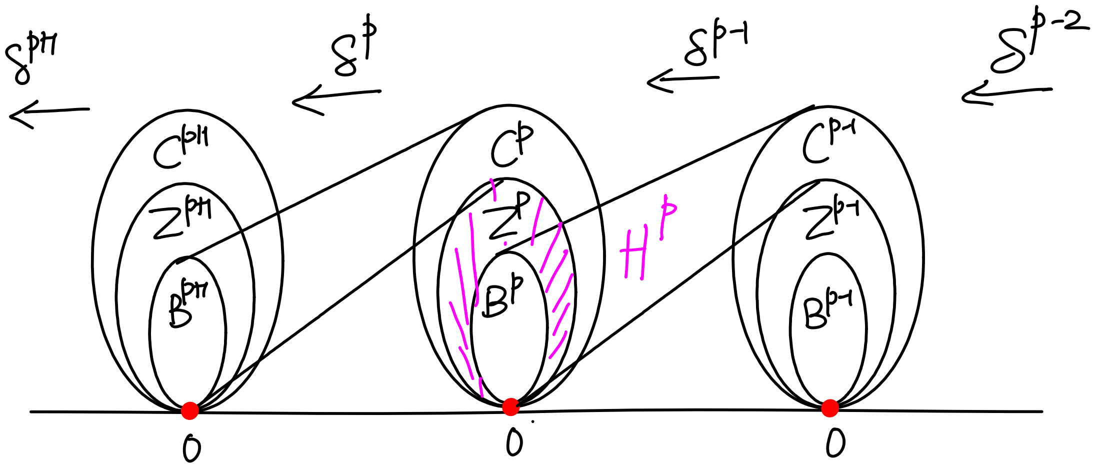
Some intuition: If ϕ evaluates a single edge to 1, and all other edges to 0, then $\delta\phi$ evaluates all triangles that are cofaces of this edge to 1, and all other triangles to 0.

We immediately get that $\delta\delta = 0$, since

$$\langle \delta\delta\phi, \bar{c}\rangle = \langle \delta\phi, \partial\bar{c}\rangle = \langle \phi, \underbrace{\partial\partial\bar{c}}_{=0}\rangle = 0.$$

Similar to $H_p = Z_p/B_p$ in homology, we can define $H^p(K;G) = Z^p(K;G)/B^p(K;G)$, the p-dimensional **cohomology group** of K with coefficients in G.

We get a complementary picture here to that of how $\{C_p, Z_p, B_p, H_p\}$ line up using $\{\partial_p\}$.



Recap:

$$C^p(K;G) = \mathrm{Hom}(C_p(K), G)$$

$$\phi^p(\bar{c}_p) = \langle \phi^p, \bar{c}_p \rangle$$

$$\langle \delta\phi, \bar{c} \rangle = \langle \phi, \partial \bar{c} \rangle, \quad \delta\delta = 0.$$

Elementary cochains

We let σ_α^* be the elementary co-chain (with $G = \mathbb{Z}$) whose value is 1 on basis element σ_α, and 0 on all other basis elements.

If $g \in G$, we let $g_\alpha \sigma_\alpha^*$ denote the cochain whose value is g_α on σ_α, and 0 on all other basis elements. We can write any p-cochain as $\phi^p = \sum g_\alpha \sigma_\alpha^*$ (possibly infinite formal sum).

With this notation, we can write down the coboundary of ϕ^p as

$$\delta \phi^p = \sum g_\alpha (\delta \sigma_\alpha^*). \qquad (*)$$