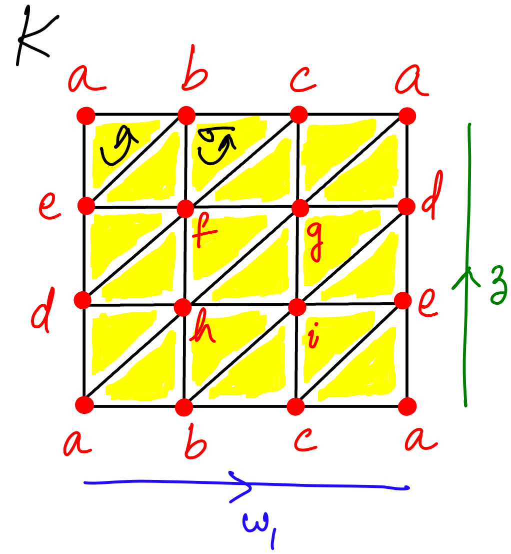
MATH 524: Lecture 27 (11/20/2025)

Today: * Cohomology of $\mathbb{K}^2$
* 0-dimensional cohomology

Example 4 (Klein bottle)



We show that $H^2(K)$ is nontrivial.

Recall, $H_2(K) = 0$.

Orient all triangles CCW. Let

$\bar{\gamma} = \sum_i f_i$ (all elementary 2-chains).

Then, $\bar{\gamma}$ is not a 2-cycle.

$$\partial \bar{\gamma} = 2\bar{z}_1, \quad \text{where} \quad \bar{z}_1 = [a,e] + [e,d] + [d,a].$$

Let σ be a 2-simplex, $[bfc]$ here. Then σ^* is a 2-cocycle (as there are no 3-simplices). Also, σ^* is not a 2-coboundary.

For, if ϕ^1 is an arbitrary 1-cochain, then

$$\langle \delta\phi^1, \bar{\gamma}\rangle = \langle \phi^1, \partial\bar{\gamma}\rangle = \langle \phi^1, 2\bar{z}_1\rangle = 2\underbrace{\langle \phi^1, \bar{z}_1\rangle}_{\text{even integer}}.$$

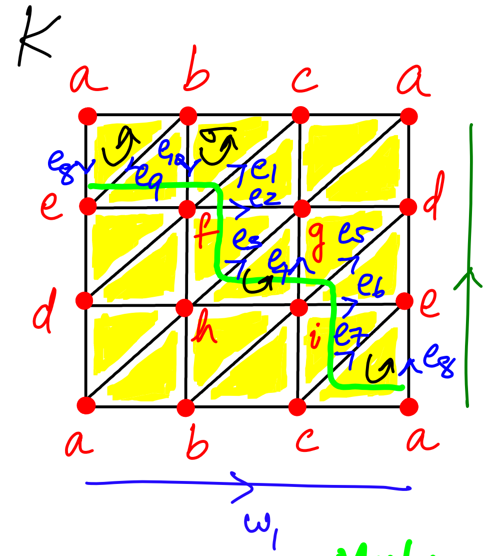
But $\langle \sigma^*, \bar{\gamma}\rangle = 1$, which is odd.

$\Rightarrow \sigma^*$ represents a nontrivial member of $H^2(K)$.

In fact, σ^* represents an element of order 2 in $H^2(K)$.

Indeed, for $\psi^1 = \sum_{i=1}^{10} e_i^*$ as shown in the figure,

$$\delta \psi^1 = 2\sigma^*.$$



The CCW orientation of σ agrees with that of both $\bar{e}_1$ and e_{10}. But all other triangles in the "band" appear twice in the expression for $\delta\psi^1$, once with +1 and once with −1 (as part of δe_i^* and δe_{i+1}^* ...).

Thus, for the Klein bottle, $H^2(K;\mathbb{Z}) \not\simeq H_2(K)$, which is yet another example where homology and cohomology groups differ in their structure.

In fact, we can show that $H^2(\mathbb{K}^2;\mathbb{Z}) \simeq \mathbb{Z}/2$.

Zero-dimensional Cohomology

Theorem 42.1 [M] $H^0(K;G)$ is the group of all 0-cochains ϕ^0 such that $\langle \phi^0, v\rangle = \langle \phi^0, w\rangle$ whenever v, w belong to the same component of $|K|$. In particular, if $|K|$ is connected, then $H^0(K) \simeq \mathbb{Z}$, and is generated by the cochain whose value is 1 on each vertex of K.

Proof $H^0(K;G)$ equals the group of 0-cocycles trivially, as there are no (-1)-dimensional simplices. If v, w are vertices that belong to the same component of $|K|$, there exists a 1-chain $\bar{c}$ of K such that $\partial \bar{c} = v - w$. Then, for any 0-cocycle ϕ^0, we have

$$0 = \langle \delta \phi^0, \bar{c}\rangle = \langle \phi^0, \partial \bar{c}\rangle = \langle \phi^0, v\rangle - \langle \phi^0, w\rangle.$$

Conversely, let ϕ^0 be a 0-cochain such that $\langle \phi^0, v\rangle - \langle \phi^0, w\rangle = 0$ whenever v, w lie in the same component of $|K|$. Then for each oriented 1-simplex σ of K,

$$\langle \delta \phi^0, \sigma\rangle = \langle \phi^0, \partial \sigma\rangle = 0.$$

So we conclude that $\delta \phi^0 = 0$. $\square$

In general, $H^0(K) \simeq$ direct product of infinite cyclic groups, one for each component of $|K|$. On the other hand, $H_0(K) \simeq$ direct sum of this collection of groups.