MATH 524: Lecture 28 (12/04/2025)

Today: * relative cohomology
* duality results

Relative Cohomology Groups

Def Let $K_0 \subseteq K$ be a subcomplex. The group of relative cochains in dimension p is defined as

$$C^p(K, K_0; G) = \operatorname{Hom}(C_p(K, K_0), G).$$

The relative coboundary, also denoted δ, is defined as the dual of the relative boundary operator:

$$\delta^p : C^p(K, K_0; G) \longrightarrow C^{p+1}(K, K_0; G).$$

We let $Z^p(K, K_0; G) = \ker \delta^p$,
$B^p(K, K_0; G) = \operatorname{im} \delta^{p-1}$, and

$$H^p(K, K_0; G) = Z^p(K, K_0; G) / B^p(K, K_0; G).$$

These are the groups of relative cocycles, relative coboundaries, and the relative cohomology group in dimension p of K modulo K_0.

While the definition is presented in a straightforward manner, the correspondence to the structure of relative homology groups is specified in a dual manner.

For chains, we have the exact sequence

$$0 \longrightarrow C_p(K_0) \xrightarrow{i} C_p(K) \xrightarrow{j} C_p(K, K_0) \longrightarrow 0$$

which splits, because $C_p(K, K_0)$ is free.

For cochains, we get a similar sequence

$$0 \longleftarrow C^p(K_0; G) \xleftarrow{\tilde{i}} C^p(K; G) \xleftarrow{\tilde{j}} C^p(K, K_0; G) \longleftarrow 0$$

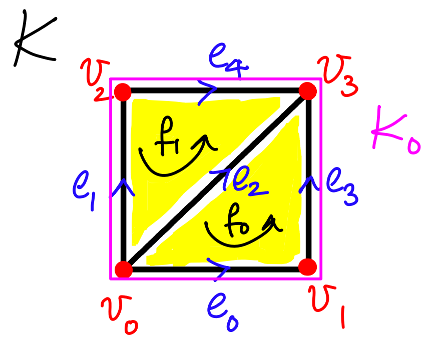
which is exact, and also splits.

$C^p(K, K_0; G)$ is a subgroup of $C^p(K; G)$ — these are the cochains that vanish on simplices carried by K_0. Equivalently, $C^p(K, K_0; G)$ is the group of cochains "carried by" $K - K_0$. Hence $\tilde{j}$ is an inclusion map.

$\tilde{i}$ is a restriction (or projection) — it is the restriction of cochain ϕ^p of $C^p(K; G)$ to simplices in K_0.

So, dual of inclusion i is projection $\tilde{i}$, and dual of projection j is inclusion $\tilde{j}$.

Examples of Relative Cohomology



1. Let K_0 consist of $\{e_0, e_1, e_3, e_4\}$ and all vertices. Let's evaluate the relative cochains.

Notice that $H_2(K,K_0) \simeq \mathbb{Z}$, $\{f_0+f_1\}$ being a generator.

f_0^*, f_1^* are relative 2-cochains; and each of them is a relative 2-cocycle (trivially, as there are no 3-simplices).

Is either of them a coboundary? No! $\delta e_1^* = -f_1^*$, $\delta e_4^* = -f_1^*$ but $e_1, e_4 \in K_0$.

e_2^* is the only relative 1-cochain. And

$\delta e_2^* = f_1^* - f_0^*$. So f_1^* and f_0^* are cohomologous.

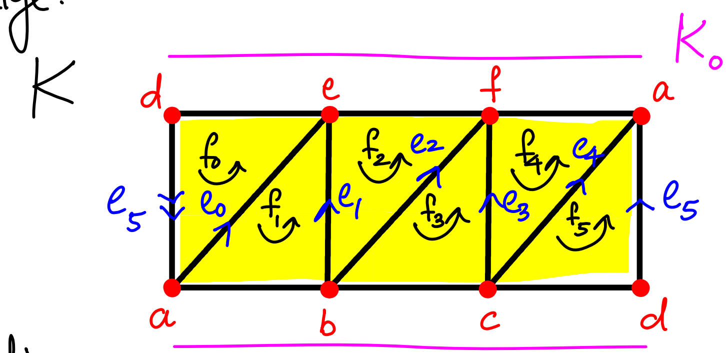
$\Rightarrow H^2(K,K_0) \simeq \mathbb{Z}$, and is generated by $\{f_0^*\}$ or $\{f_1^*\}$.

$H^1(K,K_0) = 0$, as there are no relative 1-cocycles.

$\delta e_2^* \neq 0$, $e_i^*, i = 0,1,3,4$ are trivial as those $e_i \in K_0$.

$H^0(K,K_0) = 0$, as all 0-cochains are carried by K_0.

2. Möbius strip modulo its edge.



Relative 1- and 2-cochains:

f_i^* $(i=0,\dots,5)$ are all relative 2-cochains. They are all relative 2-cocycles (trivially).

$f_0^*,\dots,f_5^*$ form a basis for $Z^2(K,K_0)$.

Similarly, $e_0^*,\dots,e_5^*$ are relative 1-cochains.

$e_0^*,\dots,e_5^*$ form a basis for $C^1(K,K_0)$.

It is convenient to use f_0^*, $f_i^*-f_{i+1}^*$, $i=0,\dots,4$ as a basis for $Z^2(K,K_0)$, and $e_0^*,\dots,e_4^*$, $e_0^*+\dots+e_5^*$ as a basis for $C^1(K,K_0)$. Then

$$\delta e_i^* = f_i^* - f_{i+1}^*, \quad i=0,\dots,4, \quad \text{and}$$

$$\delta(e_0^*+\dots+e_5^*) = 2f_0^*$$

f_0^* is not a coboundary, but $2f_0^*$ is!

$$\Rightarrow H^2(K,K_0) \simeq \mathbb{Z}/2, \quad \{f_0^*\} \text{ is a generator.}$$

$H^1(K,K_0)=0$, as there are no relative 1-cocycles $(\delta e_i^* \neq 0\ \forall i)$.

Duality

Poincaré duality §65 in [M]

Let X be a compact triangulated homology n-manifold. (generalizes the topological manifold) If X is orientable, then for each p there exists an isomorphism

$$H^p(X;G) \simeq H_{n-p}(X;G)$$

where G is an arbitrary coefficient group.

If X is nonorientable, then for each p, there exists an isomorphism

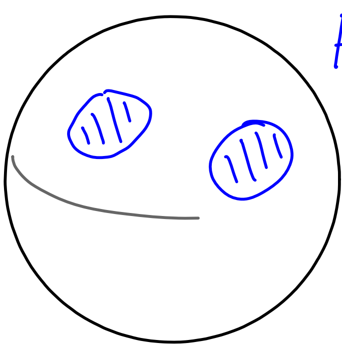
$$H^p(X;\mathbb{Z}/2) \simeq H_{n-p}(X;\mathbb{Z}/2).$$

Alexander duality §71 in [M]

Let A be a proper nonempty subset of S^n. Suppose (S^n, A) is triangulable. Then there is an isomorphism

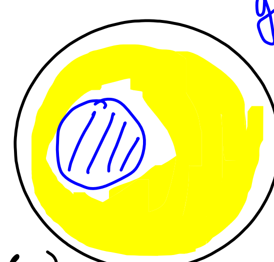
$$\tilde{H}^k(A) \simeq \tilde{H}_{n-k-1}(S^n - A)$$

S^2



A (2 discs)

$S^2 - A \cong$



stretch out one of the holes to get a disc w/ one hole.

$k=0 \Rightarrow \tilde{H}_1(S^2 - A) \simeq \mathbb{Z}$ (one hole)

$\tilde{H}^0(A) \simeq \mathbb{Z}$ (2 components).