MATH 524 : Lecture 2 (08/21/2025)

Today: * open sets, topology using open sets
* simplices, properties of simplices

We now consider topology defined in terms of open sets. This is the default approach taken in most textbooks. We first define open sets using the concept of neighborhoods.

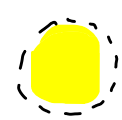
Def $\mathcal{O} \subseteq X$ is **open** if it is a neighborhood of each of its points. By (c) of **Def I**, union of any collection of open sets is also open. Also, by (b) of **Def I**, the intersection of any finite number of open sets is open.

We mention unions and finite intersections of open sets as they are both required to be open in a topology. See below.

Notice, $\mathring{N}$ (interior of neighborhood N) is always open.

Alternatively, we can start by defining open sets directly.

Def A set $A \subseteq \mathbb{R}^n$ is **open** if each $\bar{x} \in A$ can be surrounded by a ball of positive radius that lies entirely inside the set.



We can also define open sets more generally, starting with collections of subsets of some set X.

We could define neighborhoods in terms of open sets.

Def A subset $N \subseteq X$ is a neighborhood of $\bar{x}$ if there exists an open set $\mathcal{O}$ s.t. $\bar{x} \in \mathcal{O} \subseteq N$.

We now formally state the definition of topology in terms of open sets. This definition sees more use than the one using neighborhoods.

Def II A **topology** on a set X is a collection of open sets of X such that any union and finite intersection of open sets is open, and ϕ (empty set) and X are open.
The set X along with the topology is called a **topological space**.

We can define continuous functions also in terms of open sets.

Def $f: X \to Y$ is **continuous** iff (if and only if) the inverse image of each open set of Y is open in X.

We now start the discussion of homology, which is a less strict version of topological similarity than homeomorphism. We study in detail simplicial homology, where the spaces are made of "gluing" "nice" objects called simplices together, and are hence are very "regular".

As we will see, it is also much easier to algebraize questions about homology (than those about homeomorphism).

There is a "continuous" version of homology defined on spaces not composed to regular pieces (simplices), termed singular homology. It turns out singular homology is equivalent to simplicial homology.

We start by defining simplices, which are the building blocks.

Simplices

We define simplices in the usual geometric setting first, and then define them abstractly. We need some concepts from geometry first.

Def The set $\{\bar{a}_0, \ldots, \bar{a}_n\}$ of points in $\mathbb{R}^d$ is **geometrically independent** (**GI**) if for any scalars $t_i \in \mathbb{R}$, the equations

$$\sum_{i=0}^{n} t_i = 0, \quad \sum_{i=0}^{n} t_i \bar{a}_i = \bar{0} \quad \text{imply that } t_0 = t_1 = \cdots = t_n = 0.$$

Here are some observations about GI sets.

* $\{\bar{a}_i\}$ is GI $\forall i$. (singleton sets)

* $\{\bar{a}_0, \ldots, \bar{a}_n\}$ is GI $\Longleftrightarrow$ ← if and only if

$\{\bar{a}_1 - \bar{a}_0, \bar{a}_2 - \bar{a}_0, \ldots, \bar{a}_n - \bar{a}_0\}$ is linearly independent (LI).

$\bar{a}_0$ is chosen as the "origin", so to speak. But any $\bar{a}_i$ could play the role of $\bar{a}_0$ here.

IDEA: $\displaystyle\sum_{i=1}^{n} t_i(\bar{a}_i - \bar{a}_0) = \bar{0} \implies t_i = 0 \ \forall i$ (LI)

$$\sum_{i=1}^{n} t_i \bar{a}_i + \underbrace{\left(-\sum_{i=1}^{n} t_i\right)}_{t_0} \bar{a}_0 = \bar{0}$$

$\Big\}$ $\displaystyle\sum_{i=0}^{n} t_i \bar{a}_i = 0$ & $\displaystyle\sum_{i=0}^{n} t_i = 0 \implies t_i = 0 \ \forall i$

* 2 distinct points in $\mathbb{R}^d$ are GI,

3 non-collinear points are GI,

4 non-coplanar points are GI, and so on.

Notice the relationship/correspondence to LI vectors. For instance, $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ is GI, but of course the set is not LI.

Def Given GI set $\{\bar{a}_0, \dots, \bar{a}_n\}$, the **n-plane** P spanned by these points consists of all $\bar{x}$ such that

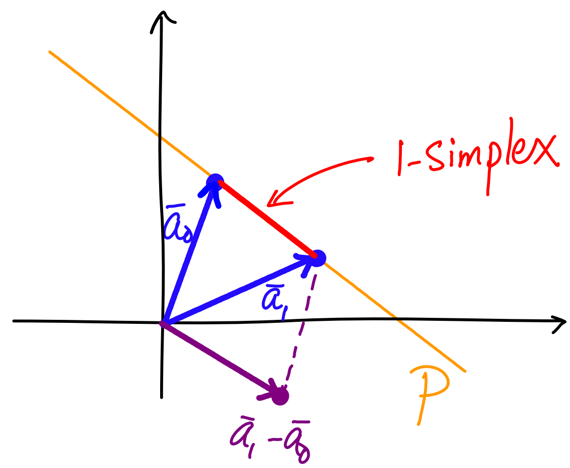
$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i \quad \text{for scalars } t_i \text{ with } \sum_{i=0}^{n} t_i = 1.$$

The scalars t_i are uniquely determined by $\bar{x}$.

Notice that t_i could be ≥ 0 or ≤ 0 here.

P can also be described as the set of $\bar{x}$ such that

$$\bar{x} = \bar{a}_0 + \sum_{i=1}^{n} t_i (\bar{a}_i - \bar{a}_0).$$



Hence P is the plane through $\bar{a}_0$ parallel to the vectors $\bar{a}_i - \bar{a}_0$.

Going back to the previous example with $\{[1\ 1], [2\ 2]\}$, the plane P is the line generated by one of the two vectors.

Q. What is the set described by $\bar{x} = \sum_{i=0}^{n} t_i a_i$, $\sum t_i = 0$?

e.g., consider $n=1$: $\bar{x} = t_0 \bar{a}_0 + t_1 \bar{a}_1$ with $t_0 + t_1 = 0 \Rightarrow t_0 = -t_1$.

$\Rightarrow \bar{x} = t_0(\bar{a}_0 - \bar{a}_1)$, i.e., it's the line generated by $\bar{a}_0 - \bar{a}_1$.

We now define a simplex as the set "spanned" by a set of GI points.

Def Let $\{\bar{a}_0, \dots, \bar{a}_n\}$ be a GI set in $\mathbb{R}^d$. The **n-simplex** σ spanned by $\bar{a}_0, \dots, \bar{a}_n$ is the set of points $\bar{x} \in \mathbb{R}^d$ s.t. $\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i$ with $\sum_{i=0}^{n} t_i = 1$, $t_i \geq 0 \; \forall i$.

The t_i are uniquely determined by $\bar{x}$, and are called the **barycentric coordinates** of $\bar{x}$ (in σ) w.r.t. $\bar{a}_0, \dots, \bar{a}_n$.

we will later extend definition of t_i to $\bar{x} \notin \sigma$. the

0-simplex : a point

1-simplex : line segment

<u>2-simplex</u>

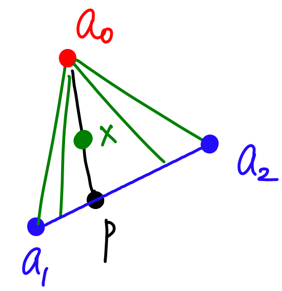
$\bar{x} = \bar{a}_0$ is trivial to consider.

Assume $\bar{x} \neq \bar{a}_0$, i.e., $t_0 \neq 1$. Now consider

$$\bar{x} = \sum_{i=0}^{2} t_i a_i = t_0 \bar{a}_0 + (1-t_0)\underbrace{\left[\frac{t_1}{1-t_0}\bar{a}_1 + \frac{t_2}{1-t_0}\bar{a}_2\right]}_{\bar{p}}$$

Since $\sum_{i=0}^{2} t_i = 1$, $1 - t_0 = t_1 + t_2$. Hence $\left(\frac{t_1}{1-t_0}\right)\bar{a}_1 + \left(\frac{t_2}{1-t_0}\right)\bar{a}_2$ is a point $\bar{p}$ on the line segment $\overline{\bar{a}_1\bar{a}_2}$, and $\bar{x} = t_0\bar{a}_0 + (1-t_0)\bar{p}$ is a point on the line segment $\overline{\bar{a}_0 \bar{p}}$.

Hence the 2-simplex is the union of such line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\overline{a_1 a_2}$, i.e., the triangle $a_0 a_1 a_2$ ($\triangle a_0 a_1 a_2$).



This result extends to higher order simplices. For instance, a tetrahedron is the union of all line segments $\overline{a_0 p}$ for all $\bar{p}$ in $\triangle a_1 a_2 a_3$.

Properties of Simplices

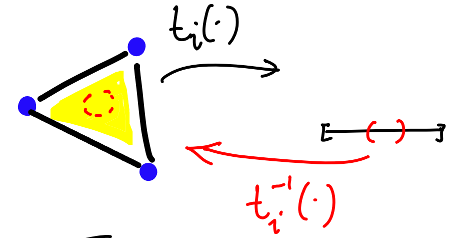
(1) $t_i(\bar{x})$ are continuous functions of $\bar{x}$.

IDEA: $t_i : \mathbb{R}^d \to \mathbb{R}$

convex hull $\{\bar{x} \mid \bar{x} = \sum_{i=0}^{n} t_i a_i, t_i \geq 0, \sum t_i = 1\}$

domain $\to$ $\mathrm{Dom}(t_i) = \mathrm{conv}(\{\bar{a}_0, \ldots, \bar{a}_n\})$

$\mathrm{Range}(t_i) = [0, 1]$



Prove that t_i^{-1}(open set in $[0,1]$) is open in σ.

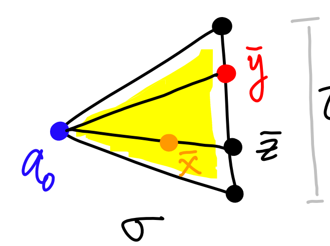
(2) σ is the union of all line segments joining $\bar{a}_0$ to points of the simplex spanned by $\{\bar{a}_1, \ldots, \bar{a}_n\}$.

Two such line segments intersect only at $\bar{a}_0$.

$\hookrightarrow$ proof?

Assume two such line segments from $\bar{a}_0$ to $\bar{y}, \bar{z} \in \tau$, the simplex spanned by $\{\bar{a}_1, \ldots, \bar{a}_n\}$, meet at $\bar{x} \neq \bar{a}_0$.

$\bar{y} \neq \vec{z}$



Then $\bar{x} = t_0 \bar{a}_0 + (1-t_0)\bar{y} = s_0 \bar{a}_0 + (1-s_0)\bar{z}$, for $t_0, s_0 \in [0,1]$,

where $t_0 \neq s_0$ by assumption (else $\bar{y} = \bar{z}$!).

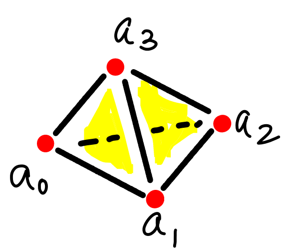
$\Rightarrow \bar{a}_0 = u\bar{y} + v\bar{z}$, where $u, v \in \mathbb{R}$ with $u+v=1$.

$\Rightarrow \bar{a}_0 \in P(\{\bar{y}, \bar{z}\}) \in P(\tau) \to$ $(n-1)$-plane spanned by $\{\bar{a}_1, \ldots, \bar{a}_n\}$.

which contradicts the GI of $\{\bar{a}_0, \ldots, \bar{a}_n\}$.

Def The points $\bar{a}_0, \dots, \bar{a}_n$ which span σ are called its **vertices**. The **dimension** of σ is n $(\dim(\sigma) = n)$. A simplex spanned by a non empty subset of $\{\bar{a}_0, \dots, \bar{a}_n\}$ is a **face** of σ. The face spanned by $\{\bar{a}_0, \dots, \hat{\bar{a}}_i, \dots \bar{a}_n\}$ where $\hat{\bar{a}}_i$ means $\bar{a}_i$ is not included, is the **face opposite $\bar{a}_i$**. Faces of σ distinct from σ itself are its **proper faces**, their union is its **boundary**, $\mathrm{Bd}\,\sigma$ or $\partial\sigma$.

$\partial(\bar{a}_0) = \phi \longrightarrow$ there are no proper faces of a vertex.



tetrahedron $a_0 a_1 a_2 a_3 = \sigma \longrightarrow$ a 3-simplex

proper faces : $\Delta a_0 a_1 a_2, \Delta a_0 a_2 a_3, \dots$ (4)

edges $\rightarrow \overline{a_0 a_1}, \overline{a_0 a_2}, \dots$ (6)

vertices $\rightarrow \bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3$ (4)

$\partial\sigma = \bigcup(\text{proper faces})$ (triangles, edges, vertices) $\longrightarrow$ the "hollow" tetrahedron

Def The **interior** of σ, $\mathrm{Int}(\sigma)$ or $\mathring{\sigma}$, is $\mathrm{Int}(\sigma) = \sigma - \mathrm{Bd}\,\sigma$.

$\mathrm{Int}(\sigma)$ is called an open simplex.

$\mathrm{Int}(\bar{a}_0) = \bar{a}_0 \longrightarrow$ as $\partial a_0 = \phi$.

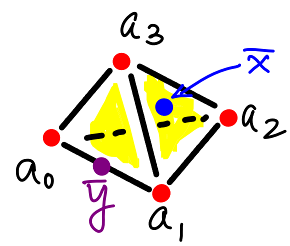
$\mathrm{Bd}\,\sigma$ consists of all $\bar{x} \in \sigma$ with at least one $t_i(\bar{x}) = 0$.

$\mathrm{Int}\,\sigma$ consists of all $\bar{x} \in \sigma$ with $t_i(\bar{x}) > 0 \ \forall i$.

could be σ itself!

Given $\bar{x} \in \sigma$, there is exactly one face τ s.t. $\bar{x} \in \operatorname{Int} \tau$.
τ is that face of σ spanned by those $\bar{a}_i$ for which $t_i(\bar{x}) > 0$.

$\bar{x}$ is interior to $\triangle a_1 a_2 a_3$
$\bar{y}$ is interior to $\overleftrightarrow{a_0 a_1}$



(3) σ is a compact, convex set in $\mathbb{R}^d$, and is the intersection of all convex sets in $\mathbb{R}^d$ containing $\bar{a}_0, \dots, \bar{a}_n$.

(4) There exists one and only one GI set of points $\{\bar{a}_0, \dots, \bar{a}_n\}$ spanning σ.

(5) $\operatorname{Int} \sigma$ is convex, and is open in P, and $Cl(\operatorname{Int} \sigma) = \sigma$. $\operatorname{Int} \sigma$ is the union of all "open line segments" joining $\bar{a}_0$ with points in $\operatorname{Int} \tau$, where τ is the face opposite $\bar{a}_0$.

closure