MATH 524: Lecture 3 (08/26/2025)

Today: * simplicial complexes
* underlying space

One more property of simplices first...

Def Unit ball: $\mathbb{B}^n = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| \leq 1\}$

Unit sphere: $\mathbb{S}^{n-1} = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x}\| = 1\}$

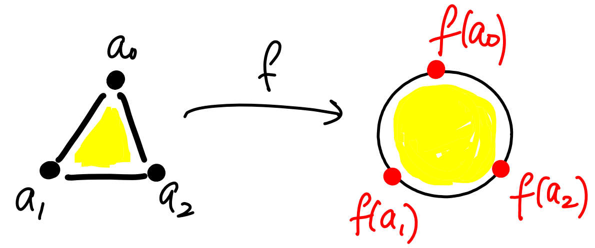
upper/lower hemisphere: $\mathbb{E}_+^{n-1} / \mathbb{E}_-^{n-1} = \{\bar{x} \in \mathbb{S}^{n-1} \mid x_n \geq 0 / x_n \leq 0\}$.

We will use these definitions later on

e.g., $\mathbb{B}^0 = \{\bar{0}\}$, $\mathbb{B}^1 = [-1, 1]$, $\mathbb{S}^0 = \{-1, 1\}$. two points

(6) There is a homeomorphism of σ (n-simplex) with $\mathbb{B}^n$ that carries $\partial\sigma$ to $\mathbb{S}^{n-1}$.

(proof in Munkres [M] EAT)



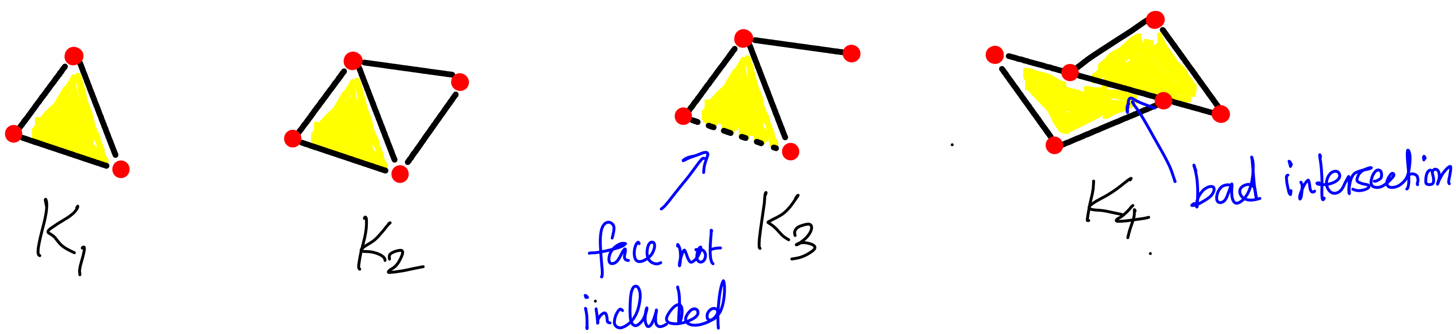
See [M] (Munkres - Elements of Algebraic Topology) for the proof.

In summary, simplices are "nice" elementary objects that can be used as building blocks to build larger spaces or objects. We will now introduce these larger objects, which are quite general, but are still "nice" since we "glue" simplices together nicely to build them.

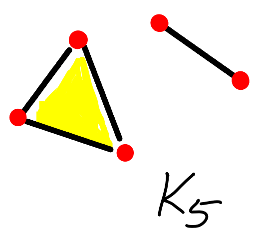
Simplicial Complexes

Def A simplicial complex K in $\mathbb{R}^d$ is a collection of simplices in $\mathbb{R}^d$ such that

(1) every face of a simplex in K is in K, and
(2) the intersection of any two simplices of K, when non-empty is a face of each of them.



K_1, K_2 are simplicial complexes, while K_3, K_4 are not.



K_5 is a simplicial complex – in particular, a simplicial complex need not be a single connected component.

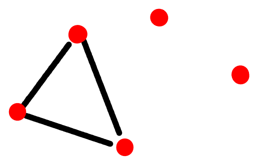
Here is another equivalent definition:

Lemma 2.1 [M] A collection of simplices K is a simplicial complex iff

(1) every face of a simplex in K is in K; and
(2) every pair of distinct simplices in K have disjoint interiors.

A simplex σ and all its proper faces together is a simplicial complex.

Def If L is a subcollection of K that contains all faces of its elements, then it is a simplicial complex on its own, called a **subcomplex** of K.

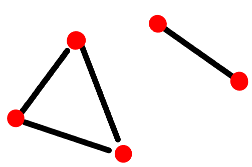


L_5.

A subcomplex of K_5

Def The subcomplex of K that is the collection of all simplices in K of dimension at most p is the **p-skeleton** of K, denoted $K^{(p)}$.

$K^{(0)}$ are the vertices of K.

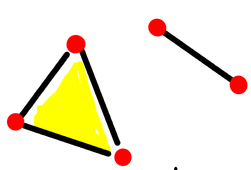


$K_5^{(1)}$ (the 1-skeleton of K_5).

Def The **dimension** of a simplicial complex K is the largest dimension of any simplex in K.

$$\dim(K) = \max_{\sigma \in K} \{\dim(\sigma)\}.$$

e.g.,



K_5

$\dim(K_5) = 2$, also referred to as a 2-complex.

A p-dimensional simplicial complex is referred to, in short, as a p-complex.

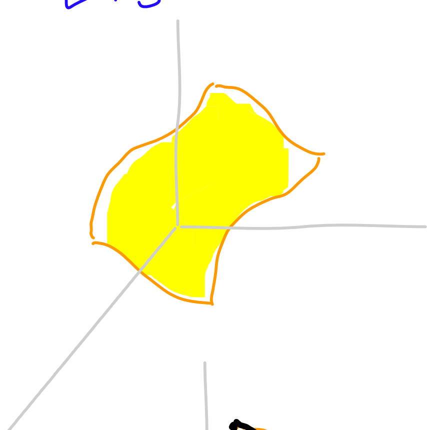
Q. What is $\dim(K^{(p)})$? → p-skeleton of K

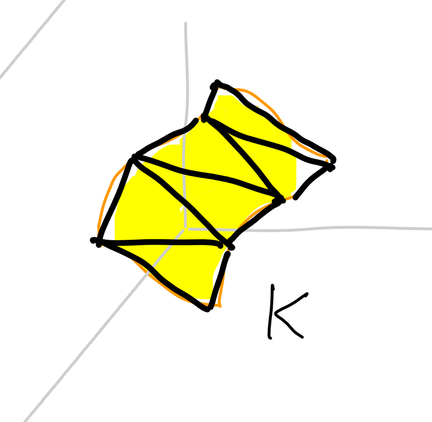
One can immediately conclude $\dim(K^{(p)}) \leq p$. But notice that $\dim(K^{(p)})$ need not always be $= p$. For instance, $\dim(K_5^{(3)}) = 2$, since $K_5^{(3)} = K_5$ itself. But if we avoid this somewhat trivial case, $\dim(K^{(p)}) = p$, typically.

Or, more generally, $\dim(K^{(p)}) = \min(p, \dim(K))$.

Recall that we want to use simplicial complexes as a "nice" structured way to model spaces. We now outline the somewhat subtle distinction between the simplicial complex and the (sub)space that it models.

Let's start with an illustration.





Consider a subspace of, say, $\mathbb{R}^3$ modeled by a sheet of paper. We could capture this space by a simplicial complex K consisting of six triangles.

Complementarily, if we start with K, we could talk about the subspace of $\mathbb{R}^3$ that it captures. We can specify the usual topology on this subspace (as inherited from $\mathbb{R}^3$).

Def Let $|K|$ be the subset of $\mathbb{R}^d$ which is the union of all simplices in K. Give each simplex its natural topology as a subspace of $\mathbb{R}^d$. Then we can topologize $|K|$ by declaring a subset A of $|K|$ is closed in $|K|$ if $A \cap \sigma$ is closed in K $\forall \sigma \in K$. $|K|$ is called the **underlying space** of K, or the **polytope** of K.

also referred to as "polyhedron"

Some people use the word polytope only when K is finite, i.e., it has a finite number of simplices, while using the word polyhedron more generally, i.e., even for the case where K is not finite.

In convex geometry, $P = \{\bar{x} \in \mathbb{R}^d \mid A\bar{x} \leq \bar{b}\}$ is a polyhedron, and a closed polyhedron is referred to as a polytope.

The two topologies — one as a subspace of $\mathbb{R}^d$, and the other defined using the simplices as above — need not be identical in all cases. But if K is finite, they usually concide. In fact, typical examples where they differ come from infinite simplicial complexes K.

$|K|$ topologized in two different ways: here is an example where the two topologies are different.

<u>Example</u> $K = \{ \bigcup [m, m+1] \ \forall m \in \mathbb{Z} \setminus \{0\} \} \bigcup$ notice that $[-1,0]$ is included

$\{ [\frac{1}{n+1}, \frac{1}{n}] \ \forall n \in \mathbb{Z}_{>0} \}$ and all faces.

K is an infinite 1-complex. infinitely many simplices

$|K| = \mathbb{R}$ as a set, but not as a topological space. Indeed,
$A = \{ \frac{1}{n}, n \in \mathbb{Z}_{>0} \}$ is closed in $|K|$, but not in $\mathbb{R}$.
A does not include 0.

But if K is finite, the topologies are the same.

<u>Properties of $|K|$</u>

Munkres – Elements of Algebraic Topology

<u>Lemma 2.2 [M]</u> If $L \subseteq K$ is a subcomplex, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.
$|\sigma|$ to be precise, but notice σ and $|\sigma|$ are identical!

<u>Lemma 2.3 [M]</u> A map $f : |K| \to X$ is continuous iff $f|_\sigma$ is continuous for each $\sigma \in K$.
f restricted to σ

Recall the barycentric coordinates of $\bar{x} \in \sigma$ ($t_{\bar{a}_i}(\bar{x})$ for vertices $\bar{a}_i$).
We can naturally extend the barycentric coordinates to $\bar{x} \notin \sigma$.

Def If $\bar{x} \in |K|$, then $\bar{x}$ is interior to precisely one simplex in K, whose vertices are, say, $\bar{a}_0, \dots, \bar{a}_n$. Then

$$\bar{x} = \sum_{i=0}^{n} t_i \bar{a}_i, \quad \text{where} \quad t_i > 0 \ \forall i, \ \sum_{i=0}^{n} t_i = 1.$$

If $\bar{v}$ is an arbitrary vertex of K, then the barycentric coordinate of $\bar{x}$ w.r.t $\bar{v}$, $t_{\bar{v}}(\bar{x})$, is defined as $t_{\bar{v}}(\bar{x}) = 0$ if $\bar{v} \notin \{\bar{a}_0, \dots, \bar{a}_n\}$, and $t_{\bar{v}}(\bar{x}) = t_i$ if $\bar{v} = \bar{a}_i$.

Notice that $t_{\bar{v}}(\bar{x})$ is continuous on $|K|$, as $t_{\bar{a}_i}(\bar{x})$ are continuous, as we noted in the last lecture, and then by Lemma 2.3.

Lemma 2.4[M] $|K|$ is Hausdorff.

A space X is Hausdorff if every pair of distinct points $\bar{x}, \bar{y} \in X$ can be surrounded by open sets $\mathcal{U}, \mathcal{V} \subseteq X$ s.t. $\bar{x} \in \mathcal{U}$, $\bar{y} \in \mathcal{V}$, $\mathcal{U} \cap \mathcal{V} = \phi$.

Proof For $\bar{x}_i \neq \bar{x}_j$ in $|K|$, by definition, there exists at least one $\bar{v}$ (vertex) s.t. $t_{\bar{v}}(\bar{x}_i) \neq t_{\bar{v}}(\bar{x}_j)$. Choose r in between $t_{\bar{v}}(\bar{x}_i)$ and $t_{\bar{v}}(\bar{x}_j)$ and define $\mathcal{U} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) < r\}$ and $\mathcal{V} = \{\bar{x} \mid t_{\bar{v}}(\bar{x}) > r\}$ as the required open sets.

We now study some important subspaces of $|K|$.