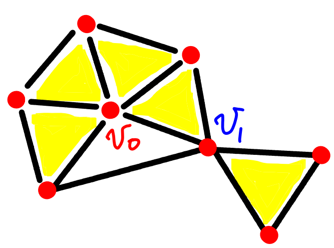
MATH 524: Lecture 4 (08/28/2025)

Today:
* star, closed star, link
* simplicial maps
* abstract simplicial complexes

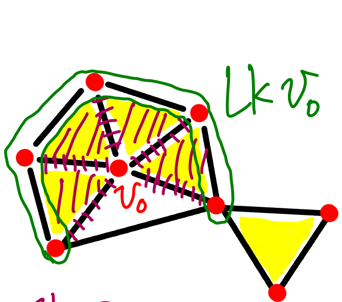
We now study some important subspaces of $|K|$.

Three Subspaces of $|K|$

Def If $\bar{v}$ is a vertex of K, then the **star** of $\bar{v}$ in K, denoted $St\,\bar{v}$ (or $St(\bar{v}, K)$) is the union of the interiors of all simplices in K that contain $\bar{v}$ as a vertex. The closure of $St\,\bar{v}$, denoted $\overline{St}\,\bar{v}$ or $Cl\,St\,\bar{v}$, is the **closed star** of $\bar{v}$. It is the union of all simplices of K which have $\bar{v}$ as a vertex. $Cl\,St\,\bar{v}$ is a polytope of a subcomplex of K. $Cl\,St\,\bar{v} - St\,\bar{v}$ is called the **link** of $\bar{v}$, denoted $Lk\,\bar{v}$.

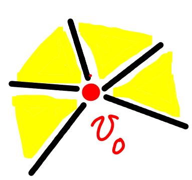
K_6 

We illustrate these subcomplexes on K_6 for vertices v_0 and v_1. Note that the unshaded triangle below v_0 is not part of K_6.

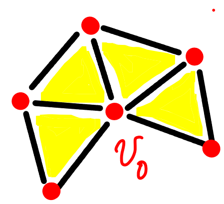


$St\,v_0$

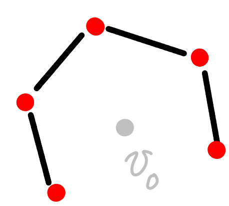
add to get $Cl\,St\,v_0$



$St\,v_0$



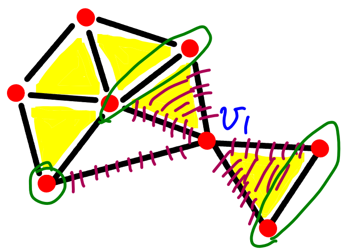
$Cl\,St\,v_0$



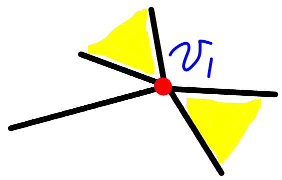
$Lk\,v_0$

Note that $Lk\,v_0 = Cl\,St\,v_0 - St\,v_0$.

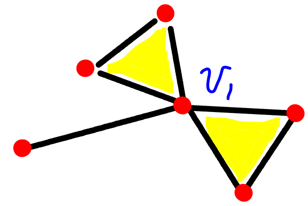
Also note that $v_0 \in St\,v_0$ (indeed, $Int\,v_0 = v_0$, and v_0 is a simplex that contains v_0 as a vertex, trivially).



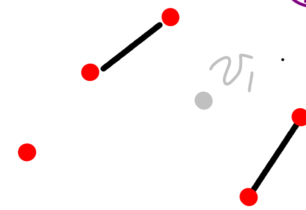
$St\, v_1$



$st\, v_1$



$Cl\,St\, v_1$



$Lk\, v_1$

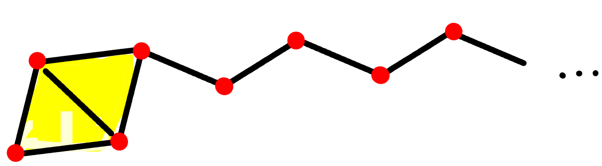
Properties of star, closed star, link

* $St\,\bar{v}$ is open in $|K|$. → We could use $t_{\bar{v}}(\cdot)$ to prove.
* The complement of $St\,\bar{v}$ is the union of all simplices that do not contain $\bar{v}$ as a vertex, and hence it is the polytope of a subcomplex of K.
* $Lk\,\bar{v}$ is the polytope of a subcomplex of K.
* $Lk\,\bar{v} = Cl\,St\,\bar{v} \cap (\text{complement of } St\,\bar{v})$.
* $St\,\bar{v}$ and $Cl\,St\,\bar{v}$ are both path-connected.
  X is path-connected if $\forall \bar{u}, \bar{v} \in X, \bar{u} \neq \bar{v}$, $\exists$ a path connecting $\bar{u}$ and $\bar{v}$ in X.
* $Lk\,\bar{v}$ need not be connected.

Def A simplicial complex K is **locally finite** if each vertex of K belongs to only finitely many simplices of K. Equivalently, K is locally finite iff each closed star is the polytope of a finite subcomplex of K.

Note: A locally finite simplicial complex could be infinite, e.g., K_7.

K_7



(the edges continue forever)

Simplicial Maps

We study maps between simplicial complexes as a first step toward developing the tools to compare spaces modeled by the simplicial complexes.

Def Let K, L be simplicial complexes. A function $f: |K| \to |L|$ is a (linear) **simplicial map** if it takes simplices of K linearly onto onto simplices of L. In other words, if $\sigma \in K$, then $f(\sigma) \in L$.

linearly: If $\sigma = \text{conv}\{\bar{v}_0, \dots, \bar{v}_n\}$ and $\bar{x} = \sum_{i=0}^{n} t_i \bar{v}_i$, $t_i \geq 0$, $\sum_{i=0}^{n} t_i = 1$, then $f(\bar{x}) = \sum_{i=0}^{n} t_i f(\bar{v}_i)$.

Note that $\{f(\bar{v}_0), \dots, f(\bar{v}_n)\}$ span a simplex τ of L, which could be of a lower dimension than σ.

Munkres takes a slightly different approach in defining simplicial maps.
[M]: Starts with $f: K^{(0)} \to L^{(0)}$, then insist that when $\{\bar{v}_0, \dots, \bar{v}_n\}$ span $\sigma \in K$, $\{f(\bar{v}_0), \dots f(\bar{v}_n)\}$ span $\tau \in L$.

f is a continuous map of σ onto τ, and hence as a map of σ onto $|L|$. Then by Lemma 2.3, it is a continuous map from $|K|$ to $|L|$.

If $g: |K| \to |L|$ and $h: |L| \to |M|$ are simplicial maps, then $f = h \circ g$ is a simplicial map from $|K|$ to $|M|$.

If we further insist that $f: K^{(0)} \to L^{(0)}$ is a **bijective** correspondence such that vertices $\bar{v}_0, \dots, \bar{v}_n$ of K span a simplex of K iff $f(\bar{v}_0), \dots, f(\bar{v}_n)$ span a simplex of L, then the induced simplicial map $g: |K| \to |L|$ is a homeomorphism. We call this map an **isomorphism** of K with L (or a simplicial homeomorphism).

Abstract Simplicial Complexes (ASCs)

Def An abstract simplicial complex (ASC) is a collection $\mathcal{S}$ of finite nonempty sets such that if $A \in \mathcal{S}$, then so is every nonempty subset of A.

Note: $\mathcal{S}$ itself could be infinite, but each $A \in \mathcal{S}$ is finite.

Example: $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ is an ASC.

We specify several more definitions related to ASCs.

Def A (any element of $\mathcal{S}$) is a **simplex** of $\mathcal{S}$. Its **dimension** is given as $\dim(A) = |A| - 1$.
↳ # elements in A, or size of A

The **dimension of the ASC** is defined as follows.

$\dim(\mathcal{S}) =$ largest dimension of any simplex in $\mathcal{S}$, or ∞ if no such largest dimension exists.

The **vertex set** V of $\mathcal{S}$ (or $V(\mathcal{S})$) is the union of all singleton elements (simplices) of $\mathcal{S}$. We do not distinguish between the individual vertices and the singleton sets they represent.

$$v_0 \text{ (vertex)} \equiv \{v_0\} \quad \text{0-simplex of } \mathcal{S}.$$

A subcollection of $\mathcal{S}$ that is a simplicial complex by itself is a **subcomplex** of $\mathcal{S}$.

We can now talk about when two ASCs are "similar".

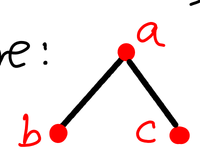
Def Two ASCs $\mathcal{S}$ and $\mathcal{T}$ are **isomorphic** if there exists a bijective correspondence f mapping $V(\mathcal{S})$ to $V(\mathcal{T})$ such that $\{a_0, \dots, a_n\} \in \mathcal{S}$ iff $\{f(a_0), \dots, f(a_n)\} \in \mathcal{T}$.

e.g., With $\mathcal{T} = \{\{d\}, \{e\}, \{f\}, \{d,e\}, \{d,f\}\}$, $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.

It turns out the previous notion of simplicial complexes (in $\mathbb{R}^d$) and ASC are directly related.

Def Let K be a (geometric) simplicial complex. Let V be its vertex set. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, \dots, a_n\}$ of V such that $\bar{a}_0, \dots, \bar{a}_n$ span a simplex of K. Then $\mathcal{K}$ is an ASC called the **vertex scheme** of K. Symmetrically, we call K a **geometric realization** of $\mathcal{K}$.

e.g., (continued) $\mathcal{S} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$ has a geometric realization S as shown here:



This complex could be sitting in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Theorem 3.1 [M] (a) Every ASC $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K.

A version of this result is given as the **geometric realization theorem** which states that every abstract d-complex has a geometric realization in $\mathbb{R}^{2d+1}$.

IDEA: If $\dim(\mathcal{S}) = d$ then let $f: V(\mathcal{S}) \to \mathbb{R}^{2d+1}$ be an injective function whose image is a set of GI points in $\mathbb{R}^{2d+1}$. Specify that for each abstract simplex $\{a_0, \dots, a_n\} \in \mathcal{S}$, $\{f(a_0), \dots, f(a_n)\} \in K$. Then $\mathcal{S}$ is isomorphic to the vertex scheme of K.