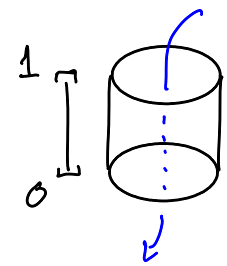
MATH 524: Lecture 5 (09/02/2025)

Today: * Examples of ASCs
* Review of algebra

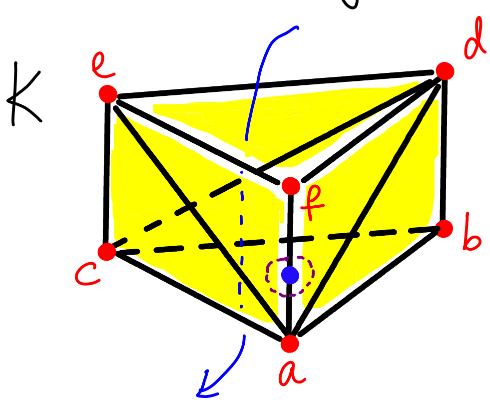
Examples of ASCs

1. Cylinder $\quad \underbrace{S^1}_{\text{circle}} \times \underbrace{I}_{[0,1]}$



We want to describe a simplicial complex K such that $|K|$ is homeomorphic to the cylinder.

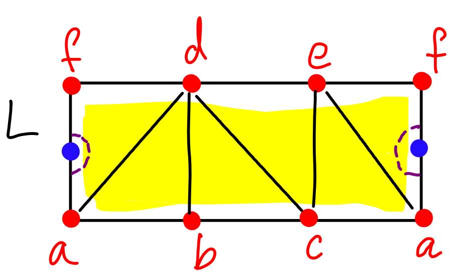
We first describe a geometric simplicial complex K, which could be sitting in $\mathbb{R}^3$, for instance.



K comprises of the six triangles adf, abd, bcd, cde, ace, and aef.

Indeed, $|K| \approx$ cylinder.

But we now specify an abstract simplicial complex whose underlying space is homeomorphic to the cylinder. We start with a rectangle L, and then assign labels to specific vertices in L. Thus, L along with the labels is the ASC.



Notice that both the left and right border edges of L are labeled af going from bottom to top.

We can describe the required map between K and L as follows.

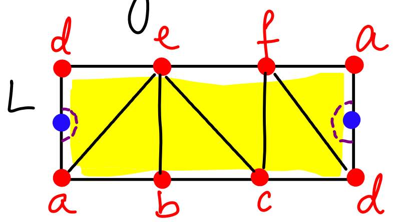
Let $f: K^{(0)} \to L^{(0)}$ is the vertex map that assigns vertices in K the labels in L. We can extend f to a simplicial map $g: |K| \to |L|$. This map g is a "pasting map", or a quotient map.

indeed, we are starting with the rectangular strip (of paper, say) L, and pasting its end edges together (af).

Notice how we can visualize a neighborhood of a point on edge af in K and correspondingly on L.

2. Möbius strip

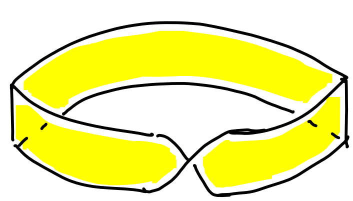
We now start with the rectangular space L and a specific vertex labeling as shown here.



The ASC $\mathcal{S}$ here has 6 triangles ade, abe, bce, cef, cdf, adf, as well as their faces.

We're again gluing the end edges, but now with a "twist."

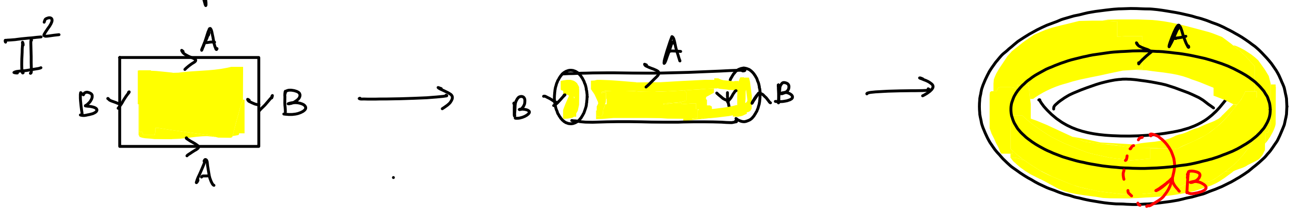
Let K be a geometric realization of $\mathcal{S}$. We can consider a simplicial map $g: |L| \to |K|$, which maps vertices in L to vertices in K. Again, g is a quotient (or "pasting") map that maps the left edge of |L| to the right edge, but with a "twist."

$L \xrightarrow{g}$ 

Notice that we do want a homeomorphism from $|L|$ to K, and just a vertex map is not enough. But of course, the vertex map is naturally (linearly) extended to the desired map from $|L|$ to $|K|$.

\mathbb(T)^2 in LaTeX!

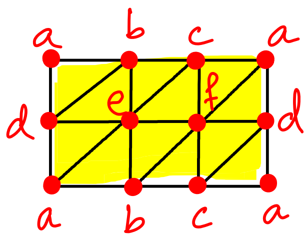
3. Torus ($\mathbb{T}^2$) The quotient space obtained by making identifications on the sides of a rectangle as follows.



Notice that this is an example of a quotient map defined on a general space, and not on an ASC.
This is the surface of a "donut", and not the solid donut itself.

Now, let us find an ASC K such that $|K| \approx \mathbb{T}^2$.
Let's start with a rectangular space as before, and assign labels that could work. Here is a first try.

K_1:

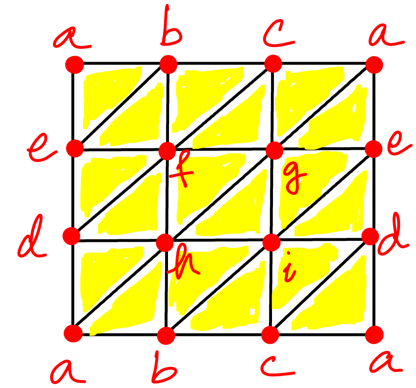


Is $|K_1| \approx \mathbb{T}^2$? No!
We are doing too much gluing!

Notice that $\overline{ad}$ is part of 4 triangles ade, adb, adc, adf, for instance. The gluings specified above glue only two edges together at a time.
With this gluing, edge ad is part of four triangles, i.e., we get a "fan" of four "flaps" meeting at ad. But notice that there are no such 4-way junctions in the torus.

We need to "spread out" more!

K_2:



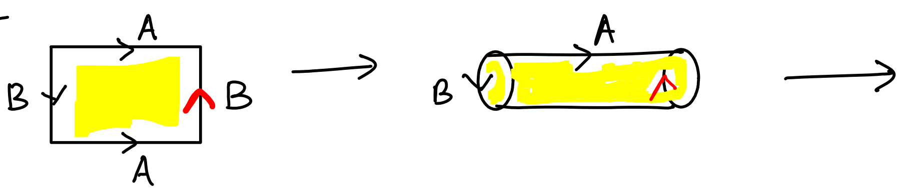
We can show that $|K_2| \approx \mathbb{T}^2$. See [M] for details, but on a complex similar to K_2.

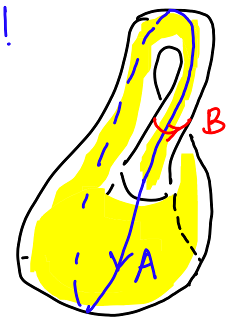
$$|K_2| \approx \mathbb{T}^2$$

Every edge is face of exactly two triangles.

4. Klein bottle ($\mathbb{K}^2$) → \mathbb{K}^2 in LaTeX!

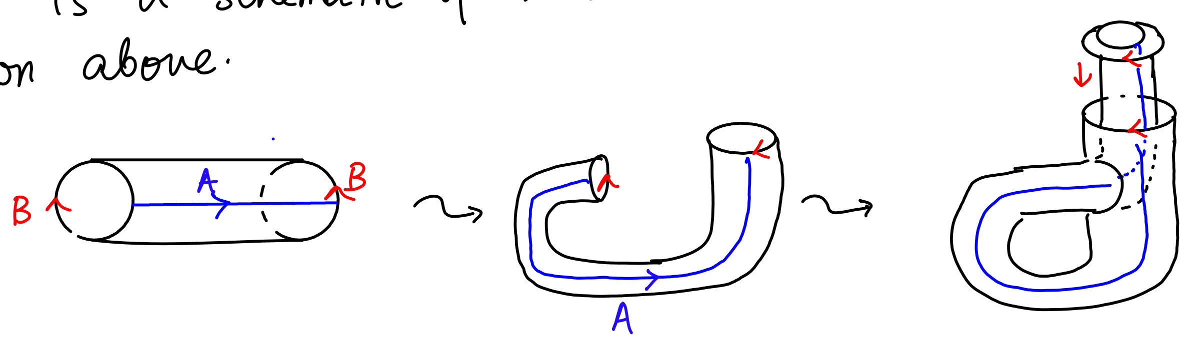
$\mathbb{K}^2$





Here, we identify the opposite pairs of edges – one pair with a twist as in the Möbius strip (B here) and the other without (A here; similar to torus or cylinder). The Klein bottle does not have an embedding in $\mathbb{R}^3$, but has in $\mathbb{R}^4$. We must go to the higher dimension to avoid self intersections.

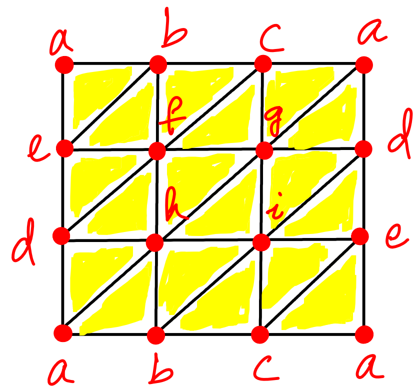
We do get an **immersion** in $\mathbb{R}^3$, which allows self intersection. Here is a schematic of how one arrives at the immersion shown above.



This instance illustrates the difficulty faced when working with geometric embeddings. We could instead work with the abstract space along with the quotient map!

We now construct an ASC for $\mathbb{K}^2$.

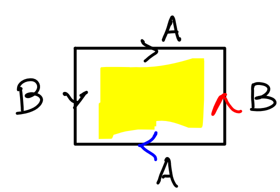
K_3



$|K_3| \approx \mathbb{K}^2 \rightsquigarrow$ one can check to make sure we are not gluing more than two edges anywhere.

Of course, $|K_3| \not\approx |K_2|$, and in general, $\mathbb{K}^2 \not\approx \mathbb{T}^2$.

Notice that we could start with the rectangular space (for L) and identify pairs of edges in several ways. For instance, when we glue both pairs of opposite edges with twists, we get the real projective plane ($\mathbb{RP}^2$).



$\mathbb{RP}^2$

The related question now is how to identify homeomorphic simplicial complexes K for any such quotient space. In particular, when do we get "nice" labelings (or gluings)?

See Lemma 3.2 in [M] for a condition given in terms of closed stars of vertices in K. This result is left as a **candidate for video tutorial**.

Review of Abelian Groups

We now review several properties and results from groups and homomorphisms between groups. The idea is to cast questions about similarity of topological spaces as corresponding questions on homomorphisms between groups defined on simplicial complexes that are homeomorphic to the spaces in question.

A good book – Fraleigh (First course in Abstract Algebra).

closure is assumed, i.e., $a+b \in G \ \forall a,b \in G$.

Group: set G with an operation $+$ "addition", such that

(1) there exists an **identity**, $0 \in G$, s.t.

$$a+0 = 0+a = a \quad \forall a \in G;$$

(2) $\forall a \in G$, there is an **inverse**, i.e., $-a \in G$ s.t.

$$a + (-a) = (-a) + a = 0; \text{ and}$$

(3) $a+(b+c) = (a+b)+c \quad \forall a,b,c \in G$; i.e., $+$ is **associative**.

(4) Further, if $a+b = b+a \ \forall a,b \in G$, then G is an **abelian group**.

In general, we will work with abelian groups in this class.

Notation: $\quad ng = \underbrace{g+g+\cdots+g}_{n \text{ times}}$ for $g \in G$.

<u>Homomorphisms</u> $f: G \to H$, G, H are groups is a homomorphism if $f(g_1 +_G g_2) = f(g_1) +_H f(g_2)$.

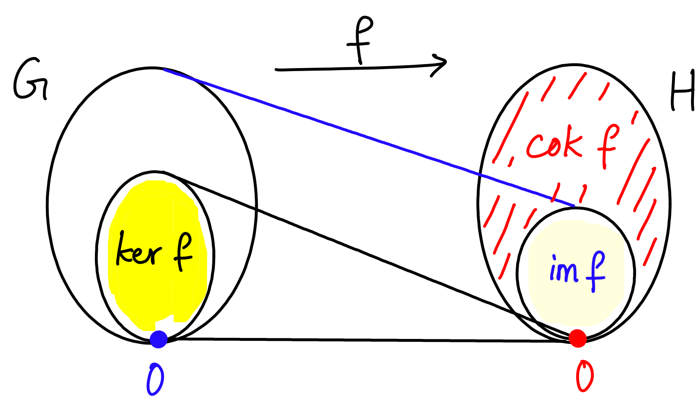
Intuitively, homomorphisms "preserve the structure" of groups.

We study subgroups specified by homomorphism f:

kernel of f: $f^{-1}(0)$, is a subgroup of G, denoted **ker f**.

image of f: $f(G)$, is a subgroup of H, denoted **im f**.

cokernel of f: quotient group of H given as $H/f(G)$, denoted **cok f**.



f is a monomorphism (injection) iff $\ker f = 0$.

f is an epimorphism (surjection) iff $\operatorname{cok} f = 0$, and in this case, f defines an isomorphism $G/\ker f \simeq H$.

An abelian group G is **free** if it has a basis $\{g_\alpha\}$ of elements in G such that $\forall g \in G$, $g = \sum n_\alpha g_\alpha$ is a unique finite sum, for $n_\alpha \in \mathbb{Z}$.

This uniqueness (for the free abelian group) implies that each basis element g_α generates an infinite cyclic group $H = \{ng_\alpha \mid n \in \mathbb{Z}\}$.

notation used in [M]

Note: $\mathbb{Z}/n$ (or $\mathbb{Z}_n$) has elements $\{0, 1, \ldots, n-1\}$ with addition mod n.

More generally, if each $g \in G$ can be written as $\sum n_\alpha g_\alpha$, but not necessarily uniquely, then $\{g_\alpha\}$ **generates** G. If $\{g_\alpha\}$ is finite, we say that G is **finitely generated**.

We will work mostly with finitely generated abelian groups

Def If G is free, and has a basis of n elements, then every basis of G has n elements. The number of elements in a basis of G is its **rank**, denoted $rk(G)$, or $rank(G)$

The **order** of G is the # elements in G, denoted $|G|$.

A crucial property: If $\{g_\alpha\}$ is a basis of G, any function f from $\{g_\alpha\}$ to abelian group H extends uniquely to a homomorphism from G to H.

Somewhat similar in flavor to a vertex map extending to the corresponding simplicial map