MATH 524: Lecture 6 (09/04/2025)

Today: * two results on abelian groups
* orientation of simplices

More results on groups...

Let G be an abelian group. $g \in G$ has **finite order** if $ng = 0$ for some $n \in \mathbb{Z}_{>0}$. The set of all elements of finite order in G is a subgroup T of G, called the **torsion subgroup**. If T vanishes, we say G is **torsion-free**.

Notice that $0 \in G$ is a trivial case in this context, as $n0 = 0$ for any $n \in \mathbb{Z}$.

We now consider how to "combine" (abelian) groups to form bigger (abelian) groups. The intuition is similar to combining multiple individual dimensions to form a higher dimensional space.

[M] defines internal direct sums, direct products, and external direct sums. We discuss them all for the sake of completeness.

Internal direct sums

Let G be an abelian group, and let $\{G_\alpha\}_{\alpha \in J}$ be a collection of subgroups of G indexed bijectively by the index set J. If each $g \in G$ can be written uniquely as finite sum $g = \sum_\alpha g_\alpha$, where $g_\alpha \in G_\alpha$ for each $\alpha \in J$, then G is the **internal direct sum** of the groups G_α, and is written $G = \bigoplus_{\alpha \in J} G_\alpha$.

If $J = \{1, 2, \ldots, n\}$ for finite n, say, we also write

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n \quad \text{or} \quad G = \bigoplus_{\alpha=1}^{n} G_\alpha$$

There is a similar distinction here to a basis vs generating set of a group.

If each $g \in G$ can be written as a finite sum $g = \sum_\alpha g_\alpha$, but not necessarily uniquely, then G is simply the <u>**sum**</u> of groups $\{G_\alpha\}$.

We write $G = \sum_\alpha G_\alpha$, or $G = G_1 + \cdots + G_n$ (if finite).

internal sum, to be precise

Here, we say $\{G_\alpha\}$ **generates** G.

Notice that if G is free abelian with basis $\{g_\alpha\}$, then G is the direct sum of subgroups $\{G_\alpha\}$, where G_α is the infinite cyclic group generated by g_α.

The converse is also true here, i.e., if G is the direct sum of $\{G_\alpha\}$ where G_α is the infinite cyclic group generated by g_α, then G is free abelian with basis $\{g_\alpha\}$.

Direct Products and External direct sums

Def Let $\{G_\alpha\}_{\alpha\in J}$ be an indexed family of abelian groups. The **direct product** $\prod_{\alpha\in J} G_\alpha$ is the group whose set is the cartesian product of sets G_α, and the operation is component-wise addition.

J can be infinite here; you could assume it is finite, though, to get the intuition. There is technical work required to extend the results and definitions to the infinite case — but it's not critical for us.

The **external direct sum** G is the subgroup of the direct product $\prod_{\alpha\in J} G_\alpha$ consisting of all tuples

$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{\alpha_i} \\ \vdots \end{pmatrix}$ such that $g_{\alpha_i} = 0_{\alpha_i}$ for all but finitely many values of α_i.

Examples

1. $G_1 = \mathbb{Z}\times\mathbb{Z}$ G_1 has rank 2; basis is $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$. $rk(G_1)=2$.

   operation is componentwise addition.

2. $G_2 = \mathbb{Z}/_2 \times \mathbb{Z}/_3$ (or $\mathbb{Z}_2\times\mathbb{Z}_3$)

   componentwise addition mod 2 and mod 3.

   $1 +_2 1 = 0$ $\quad$ $1+_3 1 = 2$, $1 +_3 2 = 0$

   G_2 is a cyclic group, $|G_2| = 6$, $\longrightarrow$ order

   $G_2 = \left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$.

   $rk(G_2) = 1$, as $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ is a basis.

   $1\times\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, $2\times\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$, $3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

   $4\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, $5\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, $6\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

Theorem The group $\prod_{i=1} \mathbb{Z}/t_i$ for $t_i \in \mathbb{Z}_{>0}$ is cyclic and is isomorphic to $\mathbb{Z}_{t_1 t_2 \cdots t_n}$ iff $\gcd(t_i, t_j) = 1 \ \forall i, j$.

t_i and t_j are relatively prime

Back to example 2: $\mathbb{Z}/2 \times \mathbb{Z}/3 \simeq \mathbb{Z}_6$.

If $n = (p_1)^{n_1}(p_2)^{n_2} \cdots (p_r)^{n_r}$ for primes $p_1, \ldots, p_r$, then

$$\mathbb{Z}_n \simeq \mathbb{Z}/(p_1)^{n_1} \times \mathbb{Z}/(p_2)^{n_2} \times \cdots \times \mathbb{Z}/(p_r)^{n_r}.$$

Structure of finitely generated abelian groups

Two main results that we will use in characterizing the structure of homology groups on simplicial complexes.

Theorem 4.2 [M] Let F be a free abelian group. If R is a subgroup of F, then R is a free abelian group. If $\operatorname{rank}(F) = n$, then $\operatorname{rank}(R) = r \leq n$. Furthermore, there is a basis $e_1, \ldots, e_n$ of F and numbers $t_1, \ldots, t_k$ ($t_i \in \mathbb{Z}_{>0}$) such that

(1) $t_1 e_1, \ldots, t_k e_k, e_{k+1}, \ldots, e_r$ is a basis for R, and

(2) $t_1 | t_2 | \cdots | t_k$, i.e., t_i divides t_{i+1} $\forall i \geq 1$. ($i \leq k-1$).

The t_i's are uniquely determined by F and R.

Intuitively, the subgroup inherits the structure of the original group...

Theorem 4.3 [M] (Fundamental theorem of finitely generated abelian groups).

Let G be a finitely generated abelian group, and let T be its torsion subgroup. The following results hold.

(a) There is a free abelian subgroup H of G such that $G = H \oplus T$. The rank of H $rk(H) = \beta$, a finite number.

(b) There exist finite cyclic groups $T_1, \ldots, T_k$ with $|T_i| = t_i > 1$, and $t_1 | t_2 | \cdots | t_k$ such that

$$T = T_1 \oplus \cdots \oplus T_k.$$

(c) The numbers β and $t_1, \ldots, t_k$ are uniquely determined by G.

β is the **betti number** of G, and $t_1, \ldots, t_k$ are the **torsion coefficients** of G.

"torsion" meaning "twistedness" or "cyclic nature"; as opposed to the free part.

A quick example on torsion...

Example What is the torsion subgroup of the multiplicative group $\mathbb{R}^*$ of all nonzero real numbers?

$G = \mathbb{R}/\{0\}$, operation is $*$ (multiplication), identity is 1, $g^{-1} = \frac{1}{g}$ $\forall g \in G$.

The answer is $\{1, -1\}$.

Here is the main consequence of the previous theorem:

Any finitely generated abelian group G can be written as a direct sum of cyclic groups, i.e.,
is isomorphic to

$$G \simeq \underbrace{(\mathbb{Z}\oplus\cdots\oplus\mathbb{Z})}_{\beta} \oplus \mathbb{Z}/t_1 \oplus \mathbb{Z}/t_2 \oplus\cdots\oplus \mathbb{Z}/t_k$$

where $\beta \geq 1$, $t_i > 1$, and $t_i \mid t_{i+1}\ \forall i$. This is a canonical form, called the **invariant factor decomposition** of G.

We can also get the **primary decomposition**, which is another canonical form:

$$G \simeq \underbrace{(\mathbb{Z}\oplus\cdots\oplus\mathbb{Z})}_{\beta} \oplus \mathbb{Z}/(p_1)^{n_1} \oplus\cdots\oplus \mathbb{Z}/(p_r)^{n_r}$$ for primes $p_1,\ldots,p_r$.

Examples

1. What are the beth number and torsion coefficients of $G = \mathbb{Z}\oplus\mathbb{Z}/4\oplus\mathbb{Z}/3\oplus\mathbb{Z}$?

$G \simeq \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/3\oplus\mathbb{Z}/4$, so $\beta = 2$.

Update! Since $\gcd(3,4)=1$ (3 and 4 are coprime), we get that $\mathbb{Z}_3\oplus\mathbb{Z}_4 \cong \mathbb{Z}_{12}$. Hence the torsion coefficient is $t_1 = 12$ here.

2. Find the primary and invariant factor decompositions of $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{18}$.

$$\begin{array}{ccc} \mathbb{Z}_4 & \mathbb{Z}_{12} & \mathbb{Z}_{18} \\ \downarrow & \downarrow & \downarrow \\ \cancel{(2)^2} & \mathbb{Z}_3 \times \mathbb{Z}_4 & \mathbb{Z}_2 \times \mathbb{Z}_9 \end{array}$$

We do not get $\mathbb{Z}_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, as 2 and 2 are not coprime.

Primary decomposition: $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9$.

invariant factor decomposition:

$$\begin{array}{ccc} 2 & 4 & 4 \\ & \times & \times \\ & 3 & 9 \\ \hline 2 & 12 & 36 \end{array}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{36}$$

Notice that $2 \mid 12 \mid 36$.

A standard "trick" is to write the factors for each prime in a line in a right justified fashion. Then multiply the numbers in each column to get the torsion coefficients.

Homology Groups

We now study groups and homomorphisms defined on simplicial complexes. Questions about topological similarity are posed as equivalent questions on corresponding groups' structure.

We need a few foundational concepts.

Orientation of a simplex

Let σ be a simplex (geometric or abstract). We define two orderings of its vertex set to be equivalent if they differ by an even permutation, i.e., you can go from one ordering to the other using an even number of pairwise swaps.

If $\dim(\sigma) > 0$, the orderings fall into two equivalence classes. Each class is an **orientation** of σ.

If $\dim(\sigma) = 0$, it has only one orientation.

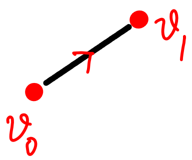
An **oriented simplex** is a simplex σ together with an orientation of σ.

Notation: Let $v_0, \ldots, v_p$ be independent. Then $\sigma = v_0 v_1 \cdots v_p$ is the simplex spanned by $v_0, \ldots, v_p$, and $[v_0, \ldots, v_p]$ denotes the oriented simplex σ with the orientation $(v_0, \ldots, v_p)$.

GI if $\bar{v}_0, \ldots, v_p \in \mathbb{R}^d$ and distinct if $v_0, \ldots, v_p$ are (just) labels in the abstract setting.

When it is clear from the context, we will use σ to denote both the simplex as well as its orientation (or the oriented simplex).

1-simplex $[v_0, v_1]$, $[v_1, v_0] \rightarrow$ opposite orientation

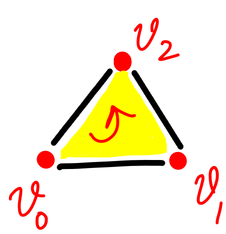


equivalent to orienting the edge from v_0 to v_1.
$[v_1 v_0] \rightarrow$ draw the arrow the other way.

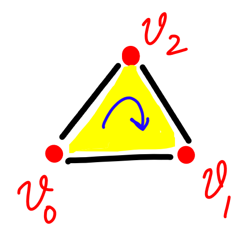
2-simplex Notice that $[v_0, v_1, v_2]$ is the same as $[v_1, v_2, v_0]$.

$$(v_0 v_1 v_2) \rightarrow (v_1 v_0 v_2) \rightarrow (v_1 v_2 v_0)$$

two pairwise swaps

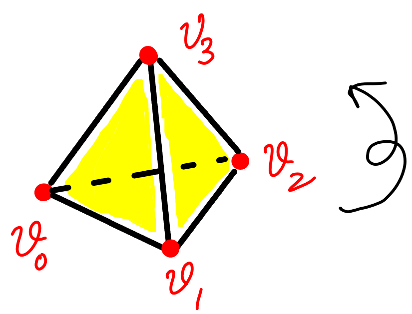


$[v_0 v_1 v_2] \rightarrow$ can be the counterclockwise orientation



$[v_0 v_2 v_1] \rightarrow$ is the clockwise orientation

3-simplex $[v_0 v_1 v_2 v_3]$



We could imagine orienting the tetrahedron as per the right-hand thumb rule – $v_0 \rightarrow v_1 \rightarrow v_2$ as the fingers of your right hand curl around, and $v_2 \rightarrow v_3$ points up along your thumb.

Notice that $[v_0 v_2 v_1 v_3]$, the opposite orientation, then corresponds to the left-hand thumb rule.